Problem Set 3
Solutions

Foundations of Number Theory

Math 435, Fall 2006

1. (20 pts.) Let p be prime, $\alpha \in \mathbb{N}$. We have $\sigma(p^\alpha) = \frac{p^\alpha - 1}{p-1} + p^\alpha < 2p^\alpha$, hence p^α is not perfect. Let q be a prime different from p, and suppose pq is perfect. Then
$$2pq = \sigma(pq) = (p+1)(q+1),$$
hence $p|(q+1)$ and $q|(p+1)$. Write $q+1 = pk$, $p+1 = ql$ with $k, l \in \mathbb{N}^{>0}$. Then $(kl)pq = (p+1)(q+1)$, hence $kl = 2$. Thus either $k = 1$, $l = 2$, or $k = 2$, $l = 1$; in the first case $p = q+1$, $2q = p+1$, hence $q = 2$, $p = 3$, and in the second case $p = 2$, $q = 3$. In both cases $pq = 6$.

2. (20 pts.) Suppose $n \in \mathbb{N}^{>0}$ is perfect. Then
$$\sum_{d|n} d = 2n,$$
hence, dividing by n on both sides of this equation:
$$\sum_{d|n} \frac{1}{n/d} = 2.$$
Now note that
$$\sum_{d|n} \frac{1}{n/d} = \sum_{d|n} \frac{1}{d}.$$

3. (20 pts.) Suppose $f * g = 0$, that is,
$$\sum_{d|n} f(d)g(n/d) = 0 \qquad \text{for all } n \in \mathbb{N}^{>0}.$$
Assume that $f, g \neq 0$; we then need to show that $f * g \neq 0$. Since $f \neq 0$, there is some $k \in \mathbb{N}^{>0}$ with $f(k) \neq 0$; take k minimal with this property. Similarly, let $l \in \mathbb{N}^{>0}$ be minimal with $g(l) \neq 0$, and put $n := kl$. We claim that $(f * g)(n) \neq 0$. To see this, we study each term in the sum
$$(f * g)(n) = \sum_{d|n} f(d)g(n/d).$$
If $d < k$ then $f(d)g(n/d) = 0$ (by minimality of k); if $d > k$ then $n/d = kl/d < l$, hence $f(d)g(n/d) = 0$ (by minimality of l). Therefore $(f*g)(n) = f(k)g(l) \neq 0$. Thus $f * g \neq 0$.

4. (20 pts.) Recall that

$$\sigma(n) = \prod_p (1 + p + p^2 + \cdots + p^{\alpha_p}),$$

where $n = \prod_p p^{\alpha_p}$ is the prime factorization of n. Suppose $\sigma(n)$ is odd. Then each of the factors $1 + p + p^2 + \cdots + p^{\alpha_p}$ is odd. If p is a prime > 2, then each of the $\alpha_p + 1$ summands in this sum is odd; hence there has to be an odd number of summands, so α_p is even. Thus n is a square (if α_2 is even) or twice a square (if α_2 is odd).

5. (5+5 pts.) Let f be a number-theoretic function.
   (a) Suppose first that g is a number-theoretic function with $f * g = \varepsilon$. Then $1 = \varepsilon(1) = (f * g)(1) = f(1)g(1)$, hence $f(1) \neq 0$. Conversely, suppose $f(1) \neq 0$. We define $g(n)$ by recursion on n. For $n = 1$ we set $g(1) := 1/f(1)$. If $n > 1$ and $g(1), \ldots, g(n-1)$ have already been defined, we put

   $$g(n) := -\frac{1}{f(1)} \sum_{\substack{d|n \\ d>1}} f(d)g(n/d).$$

   One checks immediately that then $f * g = \varepsilon$.
   (b) Suppose g, h are number-theoretic functions with $f * g = \varepsilon$ and $f * h = \varepsilon$. Then

   $$g = \varepsilon * g = (f*h)*g = f*(h*g) = f*(g*h) = (f*g)*h = \varepsilon*h = h.$$

6. (10 pts.) Let f be a multiplicative number-theoretic function with $f(1) \neq 0$. As shown in class, we have $f(1) = 1$, hence

$$(\mu f * f)(1) = \mu(1) \cdot f(1)^2 = 1 = \varepsilon(1).$$

Note also that $f^{-1}(1) = 1$ since

$$1 = \varepsilon(1) = (f^{-1} * f)(1) = f^{-1}(1) \cdot f(1).$$

Now suppose first that f is completely multiplicative. If p is a prime and $\alpha \in \mathbb{N}$, $\alpha > 0$, then

$$(\mu f * f)(p^\alpha) = \sum_{i=0}^{\alpha} \mu(p^i)f(p^i) \cdot f(p^{\alpha-i}) = f(1) \cdot f(p^\alpha) - f(p) \cdot f(p^{\alpha-1}),$$

and this equals $0 = \varepsilon(p^\alpha)$, since $f(p^\alpha) = f(p)f(p^{\alpha-1})$ by complete multiplicativity of f. Since both $\mu f * f$ and ε are multiplicative, this suffices to show $\mu f * f = \varepsilon$, hence $f^{-1} = \mu f$.— Next suppose $f^{-1} = \mu f$; then clearly for every prime p and every $\alpha > 1$ we have $f^{-1}(p^\alpha) = \mu(p^\alpha)f(p^\alpha) = 0$ since $\mu(p^\alpha) = 0$.— Finally, suppose $f^{-1}(p^\alpha) = 0$ for all prime numbers p

and $\alpha \in \mathbb{N}$, $\alpha \geq 2$. In order to check that f is completely multiplicative, it is enough to verify that $f(p^\alpha) = f(p)f(p^{\alpha-1})$ for every prime p and every $\alpha > 1$. (Why?) To see this we note that for those p and α we have

$$0 = \varepsilon(p^\alpha) = (f^{-1} * f)(p^\alpha) = \sum_{i=0}^{\alpha} f^{-1}(p^i) f(p^{\alpha-i}),$$

and this sum simplifies to $f(p^\alpha) + f^{-1}(p)f(p^{\alpha-1})$. Also

$$0 = \varepsilon(p) = (f^{-1} * f)(p) = f(p) + f^{-1}(p),$$

hence $f^{-1}(p) = -f(p)$ and thus

$$f(p^\alpha) - f(p)f(p^{\alpha-1}) = f(p^\alpha) + f^{-1}(p)f(p^{\alpha-1}) = 0,$$

so $f(p^\alpha) = f(p)f(p^{\alpha-1})$ as required.

7. (20 pts. extra credit.) Since all the functions involved are multiplicative, it is enough to show, for every prime p and every $\alpha \in \mathbb{N}$, that

$$(\tau^3 * 1)(p^\alpha) = \big((\tau * 1)(p^\alpha)\big)^2.$$

Now

$$(\tau * 1)(p^\alpha) = \sum_{i=0}^{\alpha} \tau(p^i) = \sum_{i=0}^{\alpha} i + 1 = \sum_{i=1}^{\alpha+1} i.$$

Also

$$(\tau^3 * 1)(p^\alpha) = \sum_{i=0}^{\alpha} \tau(p^i)^3 = \sum_{i=0}^{\alpha} (i+1)^3 = \sum_{i=1}^{\alpha+1} i^3.$$

So (rewriting $\alpha + 1$ as m) we have to show, for every $m > 0$, that

$$(1 + 2 + \cdots + m)^2 = 1^3 + 2^3 + 3^3 + \cdots + m^3.$$

We do this by induction on m. The base case $m = 1$ is trivial. Suppose we have shown the claim for some value of m. Then

$$\begin{gathered}\big(1 + 2 + \cdots + m + (m+1)\big)^2 = \\ (1 + 2 + \cdots + m)^2 + (m+1)^2 + 2(1 + 2 + \cdots + m)(m+1) = \\ (1^3 + 2^3 + 3^3 + \cdots + m^3) + (m+1)\big((m+1) + 2(1 + 2 + \cdots + m)\big) = \\ (1^3 + 2^3 + 3^3 + \cdots + m^3) + (m+1)\big((m+1) + m(m+1)\big) = \\ 1^3 + 2^3 + 3^3 + \cdots + m^3 + (m+1)^3,\end{gathered}$$

where in the second equality we used the inductive hypothesis, and in the third the well-known formula for $1 + 2 + \cdots + m$.

Total: 100 pts. + 20 pts. extra credit.