Problem Set 3

Due March 5

Model Theory

Math 506, Spring 2004.

Do 6 of the following problems!

1. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and A a non-empty subset of its domain. Show that the set

$$\{t^{\mathcal{M}}(a_1, \ldots, a_n): t(x_1, \ldots, x_n)\ \mathcal{L}\text{-term}, n \in \mathbb{N}, a_1, \ldots, a_n \in A\}$$

contains A and is the domain of a substructure of $\mathcal{M}$, called the **substructure of $\mathcal{M}$ generated by A** and denoted by $\langle A \rangle_{\mathcal{M}}$. Show that $\langle A \rangle_{\mathcal{M}}$ is the smallest substructure of $\mathcal{M}$ which contains A, that is, whenever $\mathcal{C}$ is a substructure of $\mathcal{M}$ whose domain C contains A as a subset, then $\mathcal{C} \supseteq \langle A \rangle_{\mathcal{M}}$.

2. Let $\mathcal{L} = \{0, 1, +, -, \cdot, <\}$ and $\mathcal{Q} = (\mathbb{Q}, 0, 1, +, -, \cdot, <)$ (the ordered field $\mathbb{Q}$).

   a) Show that for every subset S of $\mathbb{Q}$ definable in $\mathcal{Q}$ by a quantifier-free $\mathcal{L}$-formula there exists $q \in \mathbb{Q}$ such that $(q, \infty) \subseteq S$ or $(q, \infty) \cap S = \emptyset$.

   b) Use (a) to show that $\mathrm{Th}(\mathcal{Q})$ does *not* admit quantifier elimination.

3. Let K be a field and let K^{alg} be an algebraic closure of K. A polynomial $f \in K[X_1, \ldots, X_n]$ is called **absolutely irreducible** if f is irreducible when considered as an element of the polynomial ring $K^{\mathrm{alg}}[X_1, \ldots, X_n]$. (For example, the polynomial $X^2 + 1 \in \mathbb{Q}[X]$ is not absolutely irreducible, since $X^2 + 1 = (X - i)(X + i)$ in $\mathbb{Q}^{\mathrm{alg}}[X]$, whereas $XY - 1 \in \mathbb{Q}[X, Y]$ is absolutely irreducible.) For a polynomial $f \in \mathbb{Z}[X_1, \ldots, X_n]$ and a prime number p we denote by f_p the polynomial in $\mathbb{F}_p[X_1, \ldots, X_n]$ obtained by reducing the coefficients of f modulo p. Show: $f \in \mathbb{Z}[X_1, \ldots, X_n]$ is absolutely irreducible if and only if f_p is absolutely irreducible for all but finitely many primes p. (This fact is known as the Noether-Ostrowski Irreducibility Theorem.)

4. Let T be an $\mathcal{L}$-theory and $T_\forall := \{\varphi: \varphi \text{ universal sentence}, T \models \varphi\}$. Show that an $\mathcal{L}$-structure $\mathcal{A}$ is a model of $T_\forall$ if and only if there exists a model $\mathcal{M}$ of T such that $\mathcal{A} \subseteq \mathcal{M}$.

5. We say that an $\mathcal{L}$-structure $\mathcal{M}$ is **existentially closed** in an $\mathcal{L}$-structure $\mathcal{N}$ with $\mathcal{M} \subseteq \mathcal{N}$ if for every quantifier-free $\mathcal{L}$-formula $\varphi(x, y)$ with $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_m)$, $\psi(y) = \exists x(\varphi(x, y))$, and $a = (a_1, \ldots, a_m) \in M^m$, we have $\mathcal{N} \models \psi(a) \Rightarrow \mathcal{M} \models \psi(a)$. (It is enough that this holds for $n = 1$.) Let K be an infinite field and t an indeterminate over K. Show that K is existentially closed in $K[t]$ (in the language $\mathcal{L} = \{0, 1, +, \cdot, -\}$).

6. Let $(I, <)$ be a totally ordered set, $I \neq \emptyset$, and suppose that for every $i \in I$ we are given an $\mathcal{L}$-structure $\mathcal{M}_i$. We say that $(\mathcal{M}_i)_{i \in I}$ is a **chain** of $\mathcal{L}$-structures if $\mathcal{M}_i \subseteq \mathcal{M}_j$ for all $i < j$ in I. Show that there exists a unique $\mathcal{L}$-structure $\mathcal{M} := \bigcup_{i \in I} \mathcal{M}_i$ (the **union** of the chain) with the following properties: $\mathcal{M}_i \subseteq \mathcal{M}$ for all i, and if $\mathcal{N}$ is any $\mathcal{L}$-structure with $\mathcal{M}_i \subseteq \mathcal{N}$ for all i, then $\mathcal{M} \subseteq \mathcal{N}$.

7. An $\mathcal{L}$-theory T is a **$\forall\exists$-theory** if T consists only of sentences of the form $\forall x_1 \cdots \forall x_n \exists y_1 \cdots \exists y_m(\varphi)$, where $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is a quantifier-free $\mathcal{L}$-formula. Show that the union of a chain of models of a $\forall\exists$-theory T is again a model of T.