Problem Set 2

Due February 20

Model Theory

Math 506, Spring 2004.

1. Prove: if $\mathcal{F}$ is a filter on a set $I \neq \emptyset$ such that $\bigcap \mathcal{F} = \emptyset$, then every ultrafilter $\mathcal{U} \supseteq \mathcal{F}$ on I is non-principal. (Hint: use problem 6. (a) on Problem Set 1.)

2. Let $\mathcal{L}$ be a language and let T and T' be $\mathcal{L}$-theories. Suppose that for every model $\mathcal{M}$ of T there exists $\sigma \in T'$ such that $\mathcal{M} \models \sigma$. Show that there exists a finite subset $\{\sigma_1, ..., \sigma_n\}$ of T' such that $T \models \sigma_1 \vee \cdots \vee \sigma_n$.

3. Let $\mathcal{L}$ be a language and $\mathcal{M} \subseteq \mathcal{N}$ be $\mathcal{L}$-structures.

   a) Suppose that for every finite subset A of M and every $b \in N$ there exists an automorphism f of $\mathcal{N}$ which fixes A pointwise (i.e., $f(a) = a$ for all $a \in A$) and such that $f(b) \in M$. Show that then $\mathcal{M} \preccurlyeq \mathcal{N}$.

   b) Now suppose $\mathcal{L} = \{<\}$ with a binary relation symbol $<$. We consider $\mathcal{M} = (\mathbb{Q}, <)$ and $\mathcal{N} = (\mathbb{R}, <)$ as $\mathcal{L}$-structures in the natural way. Use (a) to show that $(\mathbb{Q}, <) \preccurlyeq (\mathbb{R}, <)$.

   c) Show that the converse in (a) does not hold in general. (Hint: consider $\mathcal{M} = (\mathbb{N}, <)$.)

   d) [Optional.] Let R be a commutative ring and let X and Y be infinite sets of indeterminates over R, with $X \subseteq Y$. Show that $R[X] \preccurlyeq R[Y]$, considered as structures in the language $\mathcal{L} = \{0, 1, +, \cdot\}$ of rings.

4. We say that an $\mathcal{L}$-theory T has **definable Skolem functions** if for every formula $\varphi(x_1, ..., x_n, y)$ there exists a formula $\psi(x_1, ..., x_n, y)$ such that

   a) $T \models \forall x_1 \cdots \forall x_n \exists y \psi(x_1, ..., x_n, y)$

   b) $T \models \forall x_1 \cdots \forall x_n \forall y \forall y' (\psi(x_1, ..., x_n, y) \wedge \psi(x_1, ..., x_n, y') \to y = y')$

   c) $T \models \forall x_1 \cdots \forall x_n (\exists y \varphi(x_1, ..., x_n, y) \to \exists y (\psi(x_1, ..., x_n, y) \wedge \varphi(x_1, ..., x_n, y)))$.

   In other words, in every model $\mathcal{N}$ of T, ψ defines the graph of a function $f\colon N^n \to N$ such that $\mathcal{N} \models \varphi(a, f(a))$ for all $a \in N^n$ for which $\mathcal{N} \models \exists y \varphi(a, y)$.

   a) Show that if T has built-in Skolem functions, then T has definable Skolem functions.

   b) Let $\mathcal{L} = \{0, +\}$ where $+$ is a binary function symbol and 0 is a constant symbol. Show that $T = \mathrm{Th}(\mathbb{N}, 0, +)$ has definable Skolem functions.

5. An $\mathcal{L}$-theory T is called (absolutely) **categorical** if it is satisfiable and any two models of T are isomorphic.

   a) Show that if T is categorical, then its unique model must be finite.

   b) Let $\mathcal{L} = \{f\}$ where f is a unary function symbol. Give an example of a finite $\mathcal{L}$-theory T all of whose models are infinite. (Hence T is not categorical.)

   c) Suppose that $\mathcal{L}$ is finite, and let $\mathcal{M}$ be an $\mathcal{L}$-structure whose universe is finite. Show that there exists an $\mathcal{L}$-sentence φ with the property that $\mathcal{N} \models \varphi \Longleftrightarrow \mathcal{M} \cong \mathcal{N}$ for every $\mathcal{L}$-structure $\mathcal{N}$. (In particular, $\mathcal{M} \equiv \mathcal{N} \Longleftrightarrow \mathcal{M} \cong \mathcal{N}$; thus $\mathrm{Th}(\mathcal{M})$ is categorical.)

   d) [Optional.] Show that if $\mathcal{L}$ is an arbitrary language, and $\mathcal{M}$ an $\mathcal{L}$-structure whose universe is finite, then for all $\mathcal{L}$-structures $\mathcal{N}$ we have $\mathcal{M} \equiv \mathcal{N} \Longleftrightarrow \mathcal{M} \cong \mathcal{N}$.

6. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures with $\mathcal{M} \equiv \mathcal{N}$. Show (without using the Keisler-Shelah theorem) that there exists an ultrafilter $\mathcal{U}$ on some index set I and an elementary embedding $\mathcal{N} \to \mathcal{M}^I/\mathcal{U}$.