Problem Set 4
Due Monday, Nov. 1.

*Formal Logic*

Math 430, Fall 2004

1. Let $S$ be a symbol set and let $\mathcal{M}$ be an $S$-structure, and suppose that $D$ and $E$ are subsets of $M^m$ which are definable without parameters in $\mathcal{M}$.
   (a) Show that $D \cap E$, $D \cup E$, and $M^m \setminus D$ are definable without parameters in $\mathcal{M}$.
   (b) Show that if $m > 1$, then $\pi(D) \subseteq M^{m-1}$ is definable without parameters in $\mathcal{M}$; here $\pi\colon M^m \to M^{m-1}$ is given by $\pi(a_1, \ldots, a_m) = (a_1, \ldots, a_{m-1})$.
2. Let $S = \{1, \cdot, <\}$ be a symbol set consisting of a constant symbol 1, a 2-place function symbol $\cdot$, and a 2-place relation symbol $<$. We construe $\mathbb{N}$ as an $S$-structure $\mathcal{N}$ in the usual way, by interpreting 1 by the element 1 of $\mathbb{N}$, $\cdot$ by multiplication on $\mathbb{N}$, and $<$ by the usual ordering on $\mathbb{N}$. Show that the set
$$P := \{p \in \mathbb{N} : p \text{ is prime}\}$$
of prime numbers is definable without parameters in $\mathcal{N}$.
3. The purpose of this problem is to show that the complement of a union of finitely many intervals in $\mathbb{Q}$ is also of this kind.
   (a) Let $I$ be an interval in $\mathbb{Q}$. Show that $\mathbb{Q} \setminus I$ is a union of finitely many intervals in $\mathbb{Q}$.
   (b) Let $A$ and $B$ be unions of finitely many intervals in $\mathbb{Q}$. Show that $A \cap B$ is a union of finitely many intervals in $\mathbb{Q}$.
   (c) Let $A$ be a union of finitely many intervals in $\mathbb{Q}$. Show that $\mathbb{Q} \setminus A$ is a union of finitely many intervals in $\mathbb{Q}$.
4. Let $a$ and $b$ be positive real numbers. Consider the logarithmic spiral
$$S_{a,b} := \big\{\big(ab^t \cos(t), ab^t \sin(t)\big) : t \in \mathbb{R}\big\} \subseteq \mathbb{R}^2.$$
Is $S_{a,b}$ definable with parameters in the structure
$$\mathcal{R} = (\mathbb{R}, 0^{\mathcal{R}}, 1^{\mathcal{R}}, +^{\mathcal{R}}, -^{\mathcal{R}}, \cdot^{\mathcal{R}}, <^{\mathcal{R}})?$$
(With justification.)
5. Show that $<^{\mathcal{R}}$ is not definable without parameters in the $S$-structure $(\mathbb{R}, 0^{\mathcal{R}}, +^{\mathcal{R}})$, where $S = \{0, +\}$.

(please turn)

6. Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be functions that are definable with parameters in $\mathcal{R}$. Show that there is some $a \in \mathbb{R}$ such that either
   (a) $f(t) > g(t)$ for all $t > a$, or
   (b) $f(t) = g(t)$ for all $t > a$, or
   (c) $f(t) < g(t)$ for all $t > a$.
7. (Extra credit.) Let $S = \{f\}$ be a symbol set consisting of a single 1-place function symbol $f$. Find an $S$-sentence $\varphi$ such that every $S$-structure satisfying $\varphi$ has infinite universe. Also, do the same problem for $S = \{R\}$, where $R$ is a 2-place relation symbol.