Problem Set 2
Due Friday, Sept. 24.

Formal Logic

Math 430, Fall 2004

1. Let A be a set and let $\mathcal{P}(A) := \{B : B \subseteq A\}$ be the **power set** of A (the set consisting of all subsets of A). Show that there is an injective map $\alpha\colon A \to \mathcal{P}(A)$, but that there is no surjective map $\beta\colon A \to \mathcal{P}(A)$. (Hint: assume for a contradiction that such a β exists, and consider the set $B := \{a \in A : a \notin \beta(a)\}$.)

2. Let M be a finite non-empty set and let S be a finite symbol set (i.e., only finitely many function, relation, and constant symbols). Show that there are only finitely many S-structures with universe M.

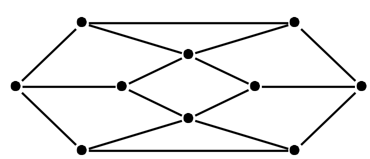
3. Let S be a symbol set, let φ, ψ be S-formulas, and let x be a variable with $x \notin \mathrm{fr}(\psi)$. Show that $\models \forall x(\varphi \wedge \psi) \leftrightarrow (\forall x\varphi \wedge \psi)$.

4. Let $S_{\text{graph}} = \{R\}$ be a symbol set with a single binary relation symbol R. An S_{graph}-structure $\mathcal{G} = (G, R^{\mathcal{G}})$ is called a **graph** if

$$\mathcal{G} \models \forall x \neg Rxx, \quad \mathcal{G} \models \forall x \forall y(Rxy \leftrightarrow Ryx).$$

The elements of G are called the **vertices** of $\mathcal{G}$. We visualize a graph $\mathcal{G} = (G, R^{\mathcal{G}})$ by thinking of its vertices as points in the plane, with vertices a and b satisfying $(a, b) \in R^{\mathcal{G}}$ connected by a line (called an **edge** of $\mathcal{G}$).

   (a) Describe



   as an S_{graph}-structure $\mathcal{G}$.

   (b) Prove or disprove: for every assignment α for $\mathcal{G}$ as in (a) we have $\mathcal{G} \models \varphi[\alpha]$, where φ is the S_{graph}-formula

$$(Rxy_1 \wedge Rxy_2 \wedge Rxy_3 \wedge Rxy_4 \to y_1 = y_2 \vee y_1 = y_3 \vee y_1 = y_4 \vee y_2 = y_3 \vee y_3 = y_4)$$

5. Consider $\mathcal{V} = (\mathbb{R}^3, \mathrm{Sp}^{\mathcal{V}})$ where Sp is a 3-place relation symbol whose interpretation in $\mathcal{V}$ is

$$\mathrm{Sp}^{\mathcal{V}} = \{(u, v, w) \in \mathbb{R}^3 : w \text{ is linearly dependent on } u \text{ and } v\}.$$

Describe in words (using linear algebra) the set

$$\{w \in \mathbb{R}^3 : ((1,0,0), (2,-1,1), w) \in \mathrm{Sp}^{\mathcal{V}}\}.$$