Math 31b : Integration and Infinite Series
Midterm 1

1. Consider the function $f(x) = x^{e^x}$.
(a) (5 points) Find the derivative $f'(x)$.

Solution. We use logarithmic differentiation. Since $\ln f(x) = \ln(x^{e^x}) = e^x \ln x$, we get

$$\frac{f'(x)}{f(x)} = (\ln f(x))' = (e^x \ln x)' = \frac{e^x}{x} + e^x \ln x$$

so

$$f'(x) = (\frac{e^x}{x} + e^x \ln x)x^{e^x}.$$

(b) (5 points) Let $g = f^{-1}$ denote the inverse function. Find $g'(1)$.

Solution. We have $f(1) = 1$, so $f^{-1}(1) = 1$ and

$$g'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(1)} = \frac{1}{(e + 1 \cdot 0)1^e} = \frac{1}{e}.$$

2. We have a sample consisting of 100 grams of a radioactive isotope. After one year it decays to 90 grams.
(a) (5 points) What is the half-life of the isotope?

Solution. The quantity must satisfy the exponential decay law $P(t) = P_0 e^{-kt}$. We have

$$P(1) = 100e^{-k} = 90$$

so the growth constant k satisfies $e^{-k} = 9/10$, i.e. $k = -\ln(9/10)$. The half-life is $-\frac{\ln 2}{\ln(9/10)}$.

(b) (5 points) After how many years will the sample decay to 30 grams?
Solution.
$P(t) = 30$ implies $100e^{-kt} = 30$ so $-kt = ln(3/10)$. Therefore $t = \frac{\ln(3/10)}{\ln(9/10)}$.

3. A function $f(x)$ satisfies the differential equation

$$f'(x) + 4f(x) = 2.$$

If $f(0) = 1$, what is $f(1)$?

Solution. We can rewrite the equation as $f' = -4(f - 1/2)$, with solution

$$f(t) = \frac{1}{2} + Ce^{-4t}$$

for some C. From $f(0) = 1$ we get $C = \frac{1}{2}$ and

$$f(1) = \frac{1 + e^{-4}}{2}.$$

4. Calculate the following limits:
(a) (5 points)

$$\lim_{x\to\pi} \frac{e^{\sin x}-1}{x-\pi}$$

Solution. It is of the form 0/0. Using l'Hôpital we get

$$\lim_{x\to\pi} \frac{e^{\sin x}-1}{x-\pi} = \lim_{x\to\pi} \frac{(\cos x)e^{\sin x}}{1} = \frac{-1\cdot e^0}{1} = -1.$$

(b) (5 points)

$$\lim_{x\to 0+} \frac{e^{-1/x^4}}{x}$$

Solution. Substitute $t = 1/x$. In terms of t, the limit is

$$\lim_{t\to\infty} te^{-t^4} = \lim_{t\to\infty} \frac{t}{e^{t^4}} = \text{(by l'Hôpital)} \lim_{t\to\infty} \frac{1}{4t^3e^{t^4}} = 0.$$

5. Calculate the definite integral:

$$\int_0^{\sqrt{\pi/4}} x^3\cos(2x^2)dx.$$

Solution.
Substitute $2x^2 = u$ so $4xdx = du$. We get

$$\int_0^{\sqrt{\pi/4}} x^3\cos(2x^2)dx = \int_0^{\pi/2} (u/2)\cos(u)\frac{du}{4} = \frac{1}{8}\int_0^{\pi/2} u\cos(u)du.$$

Using integration by parts, the last expression equals

$$\frac{1}{8}\left(u\sin(u)\Big|_0^{\pi/2} - \int_0^{\pi/2}\sin(u)du\right) = \frac{1}{8}(\frac{\pi}{2} - 0 + \cos u\Big|_0^{\pi/2}) = \frac{1}{8}(\frac{\pi}{2}-1).$$