31/B - Practice Midterm 2 - Solutions

October 28, 2011

1. (20 points) Determine whether or not the integral

$$\int_0^1 x^2 \ln x \, dx$$

converges. If it converges, compute the integral.

Solution First, we can do integration by parts:

$$\int x^2 \ln x \, dx = \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \frac{1}{x} \, dx = \frac{x^3}{3} \ln x - \int \frac{x^2}{2} \, dx = \frac{x^3}{3} \ln x - \frac{x^3}{6} + C.$$

So, we see that

$$\int_0^1 x^2 \ln x \, dx = \lim_{R \to 0^+} \frac{x^3}{3} \ln x - \frac{x^3}{6} |_R^1 = -\frac{1}{6} - \lim_{R \to 0^+} \frac{R^3}{3} \ln R = -\frac{1}{6},$$

which gives convergence and the exact value of the integral. The last limit follows, for instance, by L'Hôpital's rule.

2. (20 points) Determine whether or not the integral

$$\int_0^\infty \frac{dx}{x^2 e^{2x^3} + x^5}$$

converges. If it converges, compute the integral.

Solution This integral does not converge. For instance, we can consider the inequality

$$x^2 e^{2x^3} + x^5 = x^2(e^{2x^3} + x^3) \le x^2(e^2 + 1),$$

which is valid on the interval $[0, 1]$. Then, on this interval,

$$\frac{1}{x^2(e^2 + 1)} \le \frac{1}{x^2 e^{2x^3} + x^5}$$

But,

$$\int_0^1 \frac{dx}{x^2(e^2+1)}$$

diverges, so the integral in question also diverges.

3. (20 points) Find an N such that Simpson's rule S_N for the integral

$$\int_0^1 xe^{x^2}\,dx$$

has error of less than 10^{-9}.

Solution Consider the derivatives of $f(x) = xe^{x^2}$:

$$\begin{aligned}
f'(x) &= e^{x^2} + 2x^2e^{x^2} \\
f''(x) &= 2xe^{x^2} + 4xe^{x^2} + 4x^3e^{x^2} = 6xe^{x^2} + 4x^3e^{x^2} \\
f^{(3)}(x) &= 6e^{x^2} + 12x^2e^{x^2} + 12x^2e^{x^2} + 8x^4e^{x^2} = 6e^{x^2} + 24x^2e^{x^2} + 8x^4e^{x^2} \\
f^{(4)}(x) &= 12xe^{x^2} + 48xe^{x^2} + 48x^3e^{x^2} + 32x^3e^{x^2} + 16x^5e^{x^2} = 60xe^{x^2} + 80x^3e^{x^2} + 16x^5e^{x^2}.
\end{aligned}$$

Now, we see that $f^{(4)}(x)$ is positive and increasing on the interval $[0,1]$, so

$$|f^{(4)}(x)| \le f^{(4)}(1) = (60+80+16)e = 156e$$

for x in the interval $[0,1]$. The error bound for Simpson's Rule says that

$$Err(S_n) \le \frac{156e(1-0)^5}{180N^4} = \frac{156e}{180N^4}.$$

So, setting

$$\frac{156e}{180N^4} \le 10^{-9}$$

and solving for N, we find

$$\frac{156e10^9}{180} \le N^4.$$

Since $e \le 9$, $\frac{e}{9} \le 1$, so we can factor out $\frac{e}{9}$ and look for N satisfying

$$\frac{156 \cdot 10^9}{20} = 78 \cdot 10^8 \le N^4.$$

As $78 \le 81$, we can assume that N satisfies

$$81 \cdot 10^8 \le N^4.$$

So, $N = 3 \cdot 10^2 = 300$ works.

4. (20 points) Find the partial fraction decomposition of

$$f(x) = \frac{4x^2-20}{(2x+5)^3}.$$

Solution Let A, B, C be such that

$$\frac{4x^2-20}{(2x+5)^3} = \frac{A}{2x+5} + \frac{B}{(2x+5)^2} + \frac{C}{(2x+5)^3}.$$

Multiplying this equation by $(2x+5)^3$, we get

$$\begin{aligned} 4x^2-20 &= A(2x+5)^2 + B(2x+5) + C = A(4x^2+20x+25) + B(2x+5) + C \\ &= 4Ax^2 + (20A+2B)x + (25A+5B+C) \end{aligned}$$

By equating the coefficients, we get the following system of equations:

$$\begin{aligned} 4A &= 4 \\ 20A + 2B &= 0 \\ 25A + 5B + C &= -20 \end{aligned}$$

Thus, we see that $A = 1$, $B = -10$, and $C = 5$. Thus,

$$\frac{4x^2-20}{(2x+5)^3} = \frac{1}{2x+5} - \frac{10}{(2x+5)^2} + \frac{5}{(2x+5)^3}.$$

5. (20 points) Use Taylor polynomials and the error bound to compute the number e with an error of at most 10^{-3}.

Solution Set $f(x) = e^x$. Let $T_n(x)$ be the Taylor polynomial of e^x at $a = 0$. and let K_{n+1} be an upper bound on $f^{(n+1)}(x) = e^x$ from 0 to 1. So, we can take $K_{n+1} = 4$ (since we know that $e \leq 4$. Then, the error bound is

$$|T_n(1) - e^1| \leq \frac{4(1-0)^{n+1}}{(n+1)!} = \frac{4}{(n+1)!}.$$

So, we want $\frac{4}{(n+1)!} \leq 10^{-3}$. Consider the first few factorials:

$$\begin{aligned} 2! &= 2, \\ 3! &= 6, \\ 4! &= 24, \\ 5! &= 120, \\ 6! &= 720, \\ 7! &= 5040. \end{aligned}$$

Thus, $\frac{4}{(6+1)!} = \frac{4}{5040} = \frac{1}{1260} \leq 10^{-3}$. So, we can take $n = 6$. Then, setting

$$E = T_6(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720},$$

we know that $|E - e| \leq 10^{-3}$. We simplify this to

$$E = \frac{720 + 720 + 360 + 120 + 30 + 6 + 1}{720} = \frac{1957}{720},$$

which is our answer.