1. (10 points) Evaluate the limit, if it exists.

$$\lim_{x \to 2} \frac{x^2 - 5x + 6}{x - 2}.$$

2. (25 points) Use differentiation formulas to find the derivative of each of the following functions.

(a) $f(x) = \dfrac{\sqrt{x}}{1+x^2}$

(b) $f(x) = \sin(x \cos x)$

3. (20 points) A ball is thrown vertically upward. Its height, in meters, after t seconds is given by the function

$$S = -t^2 + 10t + 24.$$

(a) When is the ball at rest?

(b) What is the velocity of the ball when it hits the ground?

4. (25 points) Find the equation of the tangent line to the curve

$$x^3 - 9xy + y^3 = 1$$

at $(1, 3)$.

5. (20 points) Use the definition of the derivative and properties of limits to show that the derivative of

$$f(x) = \sqrt{1-x}$$

at $a = -3$ is $-\frac{1}{4}$.

MATH 31A/1 - FALL QUARTER 2007

MIDTERM EXAM

SOLUTIONS

Problem 1 Version 1 (10 points). Find the derivatives of the following five functions:

$$f_1(x) = \frac{\sin(1+x^2)}{1+x^3}\,, \qquad f_2(x) = x^{\log x}\,,$$

$$f_3(x) = (\sin x)^x\,, \qquad f_4(x) = \log(1+x+x^2)\,, \qquad f_5(x) = (\log x)^3\,.$$

Each derivative is worth 2 points. You do *not* need to simplify your answers or indicate which rule (product, quotient, chain,...) you are using. You do *not* need to indicate the domain of definition of each function.
SOLUTION.

$$\begin{aligned}
f_1'(x) &= \frac{2x\cos(1+x^2)(1+x^3) - 3x^2\sin(1+x^2)}{(1+x^3)^2}\,,\\
f_2'(x) &= \left(x^{\log x}\right)' = \left((e^{\log x})^{\log x}\right)' = (e^{(\log x)^2})' = e^{(\log x)^2}\cdot\frac{2\log x}{x} = \frac{2\log x}{x}\cdot x^{\log x}\,,\\
f_3'(x) &= \left((\sin x)^x\right)' = \left((e^{\log\sin x})^x\right)' = \left(e^{x\log\sin x}\right)'\\
&= e^{x\log\sin x}\left(\log\sin x + x\cdot\frac{\cos x}{\sin x}\right) = (\sin x)^x\left(\log\sin x + x\cdot\frac{\cos x}{\sin x}\right),\\
f_4'(x) &= \frac{1+2x}{1+x+x^2}\,,\\
f_5'(x) &= \frac{3(\log x)^2}{x}\,.
\end{aligned}$$

Problem 1 Version 2 (10 points). Find the derivatives of the following five functions:

$$f_1(x) = \frac{\sin(1+x^3)}{1+x^2}\,, \qquad f_2(x) = x^{\log x}\,,$$

$$f_3(x) = (\sin x)^x\,, \qquad f_4(x) = \log(1+x+x^2)\,, \qquad f_5(x) = (\log x)^2\,.$$

Each derivative is worth 2 points. You do *not* need to simplify your answers or indicate which rule (product, quotient, chain,...) you are using. You do *not* need to indicate the domain of definition of each function.
SOLUTION. $f_2(x)$, $f_3(x)$ and $f_4(x)$ are the same functions in Version 1. For the others we have

$$\begin{aligned}
f_1'(x) &= \frac{3x^2\cos(1+x^3)(1+x^2) - 2x\sin(1+x^3)}{(1+x^2)^2}\,,\\
f_5'(x) &= \frac{2\log x}{x}\,.
\end{aligned}$$

Problem 2 Version 1 (*a*) (5 points). Find the critical points of the function

$$f(x) = 3x^4 - 4x^3 - 12x^2\,,$$

that is, the points $\bar{x}$ where $f'(\bar{x}) = 0$. Hint for the algebra: the derivative is a polynomial of degree 3, but it has the common factor x. You do *not* need to decide whether each critical point is a maximum or minimum.
(b) (5 points). Find the maximum and the minimum of the function $f(x)$ in the interval $[-1, 3]$.
SOLUTION. (a) Equation for critical points is

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 0\,,$$

with a root $x = 0$; the other two are $-1, 2$, the roots of

$$x^2 - x - 2 = 0\,,$$

thus the critical points are $-1, 0, 2$. (b) Evaluations are

$$f(-1) = -5\,, \quad f(0) = 0\,, \quad f(2) = -32\,, \quad f(3) = 27\,,$$

accordingly $\bar{x}_m = 2$, $x_M = 3$.

Problem 2 Version 2 (a) (5 points). Find the critical points of the function

$$f(x) = 3x^4 - 4x^3 - 12x^2\,,$$

that is, the points $\bar{x}$ where $f'(\bar{x}) = 0$. Hint for the algebra: the derivative is a polynomial of degree 3, but it has the common factor x. You do *not* need to decide whether each critical point is a maximum or minimum.
(b) (5 points). Find the maximum and the minimum of the function $f(x)$ in the interval $[-2, 2]$.
SOLUTION. (a) is the same as Version 1. (b) Evaluations are

$$f(-2) = 32\,, \quad f(-1) = -5\,, \quad f(0) = 0\,, \quad f(2) = -32\,,$$

accordingly $\bar{x}_m = 2$, $x_M = -2$.

Problem 3 both versions. In what interval(s) is the function (the same function in Problem 2),

$$f(x) = 3x^4 - 4x^3 - 12x^2$$

(a) (5 points) increasing or decreasing?
(b) (5 points) convex or concave (downwards)?
Hint for the algebra: The roots of $f''(x) = 0$ contain square roots, you don't need to evaluate or simplify.
SOLUTION. (a) We can factor the derivative

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 6x(x + 1)(x - 2)\,.$$

$(a1)$ If $x \le -1$ we have $x \le 0$, $x + 1 \le 0, x - 2 \le 0$ thus

$$f'(x) \le 0 \quad \text{for } x \le -1 \Longrightarrow \ f(x) \text{ decreasing.}$$

$(a2)$ If $-1 \le x \le 0$ we have $x \le 0$, $x + 1 \ge 0$, $x - 2 \le 0$ thus

$$f'(x) \ge 0 \quad \text{for } -1 \le x \le 0 \Longrightarrow \ f(x) \text{ increasing.}$$

(a3) If $0 \le x \le 2$ we have $x \ge 0$, $x+1 \ge 0$, $x-2 \le 0$ thus

$$f'(x) \le 0 \quad \text{for } 0 \le x \le 2 \Longrightarrow f(x) \text{ decreasing.}$$

(a4) If $x \ge 2$ we have $x \ge 0$, $x+1 \ge 0$, $x-2 \ge 0$ thus

$$f'(x) \ge 0 \quad \text{for } x \ge 2 \Longrightarrow f(x) \text{ increasing.}$$

(b) We have

$$f''(x) = 36x^2 - 24x - 24 = 12(3x^2 - 2x - 2)$$

with roots

$$r_{1,2} = \frac{2 \pm \sqrt{2^2 + 4 \cdot 3 \cdot 2}}{6} = \frac{2 \pm \sqrt{28}}{6} = \frac{1 \pm \sqrt{7}}{3}$$

(this last simplification was not mandatory). We have

$$r_1 < r_2.$$

where r_1 is the root with $-$ sign, r_2 the root with the $+$ sign. We can factorize the second derivative as

$$f''(x) = 36(x - r_1)(x - r_2)\,.$$

($b1$) If $x \le r_1$ we have $x - r_1 \le 0$, $x - r_2 \le 0$ thus

$$f''(x) \ge 0 \quad \text{for } x \le r_1 \implies f(x) \text{ convex.}$$

($b2$) If $r_1 \le x \le r_2$ we have $x - r_1 \ge 0$, $x - r_2 \le 0$ thus

$$f''(x) \le 0 \quad \text{for } x \le r_1 \implies f(x) \text{ concave.}$$

($b3$) If $x \ge r_2$ we have $x - r_1 \ge 0$, $x - r_2 \ge 0$ thus

$$f''(x) \ge 0 \quad \text{for } x \le r_1 \implies f(x) \text{ convex.}$$

Problem 4 Version 1. We are trying to approximate the roots of the equation

$$f(x) = 3x^4 - 4x^3 - 12x^2 = 0 \tag{1}$$

(this is the same function in Problems 2 and 3) using bisection (parts (a) and (b)) and Newton (part (c)).
(a) (3 points). Can we start bisection in the interval $[a_0, b_0] = [-1, 3]$? If so, what is the next interval $[a_1, b_1]$?
(b) (3 points). We have done bisection 10 times, and we are taking the left endpoint a_{10} of the interval $[a_{10}, b_{10}]$ as approximation of the root $\bar{x}$. Complete the following inequality

$$|\bar{x} - a_{10}| \le b_{10} - a_{10} =$$

You *don't* need to evaluate or simplify.
(c) (4 points). Newton's iterations $\{x_n\}$ to approximate the roots of equation (1) are given by

$$x_{n+1} = g(x_n)\,.$$

What is $g(x)$? You *don't* have to simplify, or compute any of the approximations.

SOLUTION. (a) We have

$$f(-1) = -5 < 0\,, \quad f(3) = 27 > 0$$

so there is a change of sign in the interval $[-1, 3]$; bisection can be started. (b) The midpoint of the interval $[a_0, b_0] = [-1, 3]$ is $= 1$, and we have

$$f(1) = -13 < 0\,,$$

thus $[a_1, b_1] = [1, 3]$ (there is a change of sign in this interval). (b) We have

$$|\bar{x} - a_{10}| \leq b_{10} - a_{10} = \frac{b_0 - a_0}{2^{10}} = \frac{3 - (-1)}{2^{10}} = \frac{4}{2^{10}} = \frac{1}{2^8}\,.$$

(last simplification not required). (c) The function $g(x)$ is

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{3x^4 - 4x^3 - 12x^2}{12x^3 - 12x^2 - 24x}$$

(ths needed not be simplified).

Problem 4 version 2. We are trying to approximate the roots of the equation

$$f(x) = 3x^4 - 4x^3 - 12x^2 = 0 \tag{1}$$

(this is the same function in Problems 2 and 3) using bisection (parts (a) and (b)) and Newton (part (c)).

(a) (3 points). Can we start bisection in the interval $[a_0, b_0] = [1, 3]$? If so, what is the next interval $[a_1, b_1]$?

(b) (3 points). We have done bisection 10 times, and we are taking the left endpoint a_{10} of the interval $[a_{10}, b_{10}]$ as approximation of the root $\bar{x}$. Complete the following inequality

$$|\bar{x} - a_{10}| \leq b_{10} - a_{10} =$$

You *don't* need to evaluate or simplify.

(c) (4 points). Newton's iterations $\{x_n\}$ to approximate the roots of equation (1) are given by

$$x_{n+1} = g(x_n)\,.$$

What is $g(x)$? You *don't* have to simplify, or compute any of the approximations.

SOLUTION. (a) We have

$$f(1) = -13 < 0\,, \quad f(3) = 27 > 0$$

so there is a change of sign in the interval $[1, 3]$; bisection can be started. (b) The midpoint of the interval $[a_0, b_0] = [1, 3]$ is $= 2$, and we have

$$f(2) = -32 < 0$$

thus $[a_1, b_1] = [2, 3]$ (there is a change of sign in this interval). (b) We have

$$|\bar{x} - a_{10}| \leq b_{10} - a_{10} = \frac{b_0 - a_0}{2^{10}} = \frac{3 - 1}{2^{10}} = \frac{2}{2^{10}} = \frac{1}{2^9}\,.$$

(last simplification not required). (c) same as Version 1.

1. Use differentiation formulas to find the derivative of each of the following functions.

(a) $y = \dfrac{x^3}{\cos x}$

(b) $y = \sqrt{x} + 1 + \dfrac{1}{\sqrt{x}}$

2. If f is the focal length of a convex lens and an object is placed at a distance p from the lens, then its image will be at a distance q from the lens, where f, p and q are related by the ***lens equation***

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}.$$

If the focal length of the lens is 5 cm, how fast is the distance of the image from the lens changing when the object is 30 cm from the lens?

3. Evaluate each limit, if it exists.

(a) $\lim_{r \to 2} \dfrac{r^2 + 4}{r - 1}$

(b) $\lim_{t \to 0} \dfrac{\sqrt{2 - t} - \sqrt{2}}{t}$

4. Find the equations of all tangent lines to the curve $y = \frac{1}{3}x^3 - x^2 + 2$ that are parallel to the line $y = 3x - 4$.

5. Use the definition of the derivative to show that the derivative of $f(x) = \cos x$ is $f'(x) = -\sin x$. You will need the trigonometric identity of the form

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

and limit theorems from the text as well as limit laws.

1. Differentiate

$$(a) \qquad f(x) = \frac{2x+1}{1+\sec x}$$

$$(b) \qquad f(x) = \sqrt{x - 2\sqrt{x^2-1}}$$

(a) $f'(x) = \dfrac{2(1+\sec x) - (2x+1)(\sec x \tan x)}{(1+\sec x)^2}$

(b) $f(x) = (x - 2(x^2-1)^{1/2})^{1/2}$

$f'(x) = \frac{1}{2}(x-2(x^2-1)^{1/2})^{-1/2}(1-2(\frac{1}{2})(x^2-1)^{-1/2}(2x))$

2. Find all the values of x for which the tangent lines to the graph of

$$f(x) = x^2 - x + 1$$

at $(x, f(x))$ pass through the origin.

$$y - 0 = m(x - 0)$$

$$m = f'(x) = 2x - 1$$

$$y = x^2 - x + 1$$

$$x^2 - x + 1 = (2x-1)(x) = 2x^2 - x$$

$$-x^2 + 1 = 0 \qquad x = +1, -1$$

3. Evaluate the limits, if they exist.

$$(a) \qquad \lim_{x\to 2} \frac{x^3 - 3x^2 + 2x}{x^2 - x - 2}$$

$$(b) \qquad \lim_{x\to 0} x \cot x$$

$$(a)\ \lim_{x\to 2} \frac{x(x^2-3x+2)}{x^2-x-2} = \lim_{x\to 2} \frac{x\cancel{(x-2)}(x-1)}{\cancel{(x-2)}(x+1)}$$

$$= \frac{2(2-1)}{2+1} = 2/3$$

$$(b)\ \lim_{x\to 0} \frac{x\cos x}{\sin x} = \lim_{x\to 0} \frac{\cos x}{\frac{\sin x}{x}}$$

$$= \frac{\lim_{x\to 0} \cos x}{\lim_{x\to 0} \frac{\sin x}{x}} = \frac{1}{1} = 1$$

4. Find the equation of the tangent line to the curve

$$(x^2 + y^2)^2 = 2(x^2 y + 1)$$

at $(-1, 1)$.

$$\cancel{2}(x^2+y^2)(2x+2y\frac{dy}{dx}) = \cancel{2}(2xy + x^2\frac{dy}{dx})$$

$$(1+1)(-2+2\frac{dy}{dx}) = -2 + \frac{dy}{dx}$$

$$-4+4\frac{dy}{dx} = -2\frac{dy}{dx} \qquad \frac{dy}{dx} = 2/3$$

$$y-1 = \frac{2}{3}(x+1)$$

5. Use the **definition** of the derivative to find the derivative of

$$f(x) = \sqrt{2x^2+1}$$

$$f'(a) = \lim_{x \to a} \frac{\sqrt{2x^2+1} - \sqrt{2a^2+1}}{x-a} \cdot \frac{\sqrt{2x^2+1} + \sqrt{2a^2+1}}{\sqrt{2x^2+1} + \sqrt{2a^2+1}}$$

$$= \lim_{x \to a} \frac{(2x^2+1) - (2a^2+1)}{(x-a)(\sqrt{2x^2+1} + \sqrt{2a^2+1}}$$

$$= \lim_{x \to a} \frac{2(x^2-a^2)}{(x-a)(\sqrt{2x^2+1} + \sqrt{2a^2+1})}$$

$$= \lim_{x \to a} \frac{2\cancel{(x-a)}(x+a)}{\cancel{(x-a)}(\sqrt{2x^2+1} + \sqrt{2a^2+1})}$$

$$= \frac{2(2a)}{2\sqrt{2a^2+1}}$$

Problem 1. Let $f(x)=\sqrt{x}$. Using the definition of the derivative prove that

$$f'(x)=\frac{1}{2\sqrt{x}}.$$

Solution. The function $f(x)$ is only defined when $x \geq 0$, so we will assume that $x \geq 0$ for the remainder of the solution.

By the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}. \end{aligned}$$

We multiply the numerator and denominator by the conjugate of $\sqrt{x+h}-\sqrt{x}$:

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{(\sqrt{x+h}-\sqrt{x})\cdot(\sqrt{x+h}+\sqrt{x})}{h\cdot(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0}\frac{(\sqrt{x+h})^2-(\sqrt{x})^2}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0}\frac{h}{h}\cdot\frac{1}{\sqrt{x+h}+\sqrt{x}} \\ &= \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}. \end{aligned}$$

Since the square root function is continuous, $\sqrt{x+h}\to\sqrt{x}$ as $h\to 0$, so that as $h\to 0$, $\sqrt{x+h}+\sqrt{x}\to 2\sqrt{x}$. Thus if $x=0$, the limit in the definition of the derivative does not exist. If $x>0$, the quotient rule for limits gives us that the limit in the definition of $f'(x)$ is exactly

$$f'(x)=\frac{1}{2\sqrt{x}}.$$

$\square$

Problem 2. Given that

$$\lim_{h\to 0}\frac{\sin h}{h}=1,$$

prove, using the definition of the derivative, that if $f(x)=\sin x$, then $f'(x)=\cos x$.

Solution. Let $f(x)=\sin x$. By the definition of the derivative,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\frac{\sin(x+h)-\sin x}{h}.$$

Using the formula $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$, we find

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin x\cos h + \sin h\cos x - \sin x}{h} \\ &= \lim_{h\to 0} \left(\frac{\sin h}{h}\cos x + \frac{\cos h - 1}{h}\sin x \right). \end{aligned}$$

Since $\frac{\sin h}{h} \to 1$ as $h \to 0$, we get (using additivity of limits) that

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin h}{h}\cos x + \lim_{h\to 0} \frac{\cos h - 1}{h}\sin x \\ &= \cos x + \lim_{h\to 0} \frac{\cos h - 1}{h}\sin x. \end{aligned}$$

Using the half-angle formula

$$\sin^2\frac{\alpha}{2} = \frac{1}{2}(1-\cos\alpha),$$

we can rewrite this as

$$\begin{aligned} f'(x) &= \cos x + \lim_{h\to 0} \frac{2\sin^2\frac{h}{2}}{h} \\ &= \cos x + \lim_{h\to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}} \cdot \sin\frac{h}{2}. \end{aligned}$$

Since $\frac{h}{2} \to 0$ as $h \to 0$, we find that $\lim_{h\to 0} \frac{\sin(h/2)}{(h/2)} = \lim_{h\to 0} \frac{\sin h}{h} = 1$. Thus, using multiplicativity of limits, we can write

$$\begin{aligned} f'(x) &= \cos x + \left(\lim_{h\to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}} \right) \cdot \left(\lim_{h\to 0} \sin\frac{h}{2} \right) \\ &= \cos x + 1 \cdot \lim_{h\to 0} \sin\frac{h}{2}. \end{aligned}$$

Since $\sin\frac{h}{2}$ is a continuous function, its limit at 0 is $\sin 0 = 0$. We therefore conclude that

$$f'(x) = \cos x.$$

□

Problem 3. Find $\frac{dy}{dx}$ given that: (a) $y = 4x^2$; (b) $y = \sin(x) + \sin(2x^2)\cdot\cos(3x^3)$; (c) $y = \frac{\sin(x)+1}{\cos(x)+2}$; (d) $y = \sec(\tan(x+y))$.

Solution. (a)

$$\frac{d(4x^2)}{dx} = \frac{4d(x^2)}{dx} = 4\cdot 2x = 8x.$$

(b)

$$
\begin{aligned}
\frac{d(\sin x + (\sin 2x^2)\cdot(\cos(3x^3))}{dx} &= \frac{d\sin x}{dx} + \frac{d((\sin 2x^2)(\cos(3x^3))}{dx} \\
&= \cos x + \frac{d(\sin 2x^2)}{dx}\cos 3x^3 + \left(\sin 2x^2\right)\frac{d(\cos 3x^3)}{dx} \\
&= \cos x + \cos(2x^2)\frac{d(2x^2)}{dx}\cos 3x^3 \\
&\quad + \left(\sin 2x^2\right)(-\sin(3x^3))\frac{d(3x^3)}{dx} \\
&= \cos x + \cos(2x^2)4x(\cos 3x^3) - \sin(2x^2)\sin(3x^3)9x^2 \\
&= \cos x + 4x\cos(2x^2)\cos(3x^3) - 9x^2\sin(2x^2)\sin(3x^3).
\end{aligned}
$$

(c)

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{d(\sin x+1)}{dx}(\cos x+2) - \frac{d(\cos x+2)}{dx}(\sin x+1)}{(\cos x+2)^2} \\
&= \frac{(\cos x)(\cos x+2) - (-\sin x)(\sin x+1)}{(\cos x+2)^2} \\
&= \frac{\cos^2 x + 2\cos x + \sin^2 x + \sin x}{(\cos x+2)^2} \\
&= \frac{1+2\cos x+\sin x}{(\cos x+2)^2}.
\end{aligned}
$$

(d) We use implicit differentiation:

$$
\begin{aligned}
\frac{dy}{dx} &= \sec'(\tan(x+y))\cdot\tan'(x+y)\cdot\frac{d(x+y)}{dx} \\
&= \sec(\tan(x+y))\tan(\tan(x+y))\cdot\sec^2(x+y)\cdot(1+\frac{dy}{dx}) \\
&= \sec(\tan(x+y))\tan(\tan(x+y))\sec^2(x+y) \\
&\quad + \frac{dy}{dx}(\sec(\tan(x+y))\tan(\tan(x+y))\sec^2(x+y)).
\end{aligned}
$$

Thus, by moving the terms involving dy/dx to the left side of the equation, we get

$$\frac{dy}{dx}(1-\sec(\tan(x+y))\tan(\tan(x+y))\sec^2(x+y)) = \sec(\tan(x+y))\tan(\tan(x+y))\sec^2(x+y).$$

Solving this we get

$$\frac{dy}{dx} = \frac{\sec(\tan(x+y))\tan(\tan(x+y))\sec^2(x+y)}{1-\sec(\tan(x+y))\tan(\tan(x+y))\sec^2(x+y)}.$$

$\square$

Problem 4. Find a point where the curve $y = x^3 + 3x^2 + 3x + 5$ has a horizontal tangent.

Solution. The curve has a horizontal tangent at the points where $\frac{dy}{dx}$ is zero.

We find

$$\frac{dy}{dx} = 3x^2 + 6x + 3 = 3(x^2 + 2x + 1) = 3(x+1)^2.$$

This expression is zero when $x = -1$. The corresponding value of y is $-1+3-3+5 = 4$. Therefore, the curve has a horizontal tanget at the point $(-1,4)$. $\square$

Problem 5. A trough is 10 feet long and its ends have the shapes of isosceles triangles that are 3ft across at the top and have a height of 1 foot. If the trough is filled with water at a rate of 12 ft^3/min, how fast is the water level rising when the water is 9 inches deep.

Solution. Let $V(t)$ be the volume of the water in the trough at a given time t, and let $h(t)$ be the height of the water level at the same time. Then $V'(t)$ is the rate at which the tank is filled, and $h'(t)$ is the rate at which the water level is rising. Thus we are given $V'(t)$ and we need to find $h'(t)$.

If the water level is $h(t)$, then the amount of water the trough holds is given by its length (10ft) times the area of the part of the end which is covered with water. The shape of this part is an isosceles triangle with height h, which is similar to the given triangle (whose height is 1ft and width is 3ft). Thus the desired area is $h(t)^2$ times the area of the end of the trough, which is $\frac{1}{2} 1 \cdot 3 = 3/2$. Thus

$$V(t) = 10 \cdot \frac{3}{2} \cdot h^2(t) = 15h^2(t).$$

Differentiating this equation implicitely in t we get

$$V'(t) = 15\frac{dh^2(t)}{dt} = 15 \cdot 2h(t) \cdot h'(t) = 30h(t)h'(t).$$

Solving for $h'(t)$ gives

$$h'(t) = \frac{V'(t)}{30h(t)}.$$

We are given that at the time we are interested in, $V'(t) = 12$ and $h(t)$ is 9 inches (i.e., $9/12 = 3/4$ of a foot). Thus

$$h'(t) = \frac{12}{30\frac{3}{4}} = \frac{8}{15}.$$

$\square$

Problem 6. Let C be the curve defined by $x^3 + xy + y^3 = 3$ and which goes through the point $(1,1)$. What is the slope fo the tangent line to C at $(1,1)$?

Solution. We implicitely differentiate the quation for C:

$$3x^2 + y + x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0.$$

Thus

$$\frac{dy}{dx}(x + 3y^2) = -3x^2 - y.$$

Thus
$$\frac{dy}{dx} = \frac{-3x^2 - y}{x + 3y^2}.$$
At the given point $x = 1$ and $y = 1$. Substituting these in we get
$$\frac{dy}{dx} = -1.$$
Thus the slope is -1. □

Problem 7. Give an example of a function that is continuous on $I = [-1, 1]$ but not differentiable at at least two points in $(-1, 1)$.

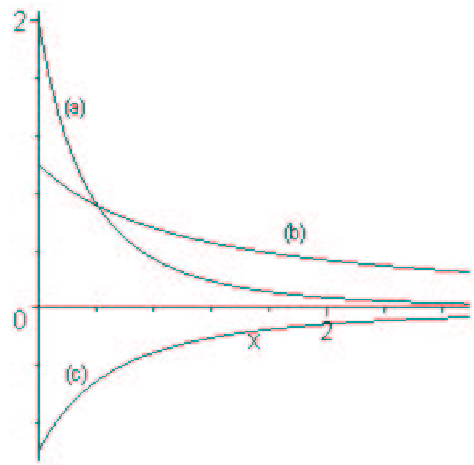
Solution. Let $f(x) = |x + \frac{1}{2}||x - \frac{1}{2}|$. We claim that the function is not differenentiable at $-\frac{1}{2}$ and $\frac{1}{2}$, but is continuous.

The function $f(x)$ is a continuous function, because the functions $|x + \frac{1}{2}|$ and $|x - \frac{1}{2}|$ are continuous, and therefore so is their product, $f(x)$.

To see that $f(x)$ is not differentiable at $\frac{1}{2}$, we compute its derivative at that point using the definition of the derivative:
$$\begin{aligned} f'(\frac{1}{2}) &= \lim_{h \to 0} \frac{|(\frac{1}{2} + h) + \frac{1}{2}||(\frac{1}{2} + h) - \frac{1}{2}| - |\frac{1}{2} + \frac{1}{2}||\frac{1}{2} - \frac{1}{2}|}{h} \\ &= \lim_{h \to 0} \frac{|h + 1||h|}{h}. \end{aligned}$$
Since $|x|$ is a continuous function, $\lim_{h \to 0} |h + 1| = |1| = 1$. Using the product rule for limits, we therefore conclude that
$$f'(\frac{1}{2}) = \lim_{h \to 0} |h + 1| \cdot \lim_{h \to 0} \frac{|h|}{h} = \lim_{h \to 0} \frac{|h|}{h}.$$
But this limit does not exist. Indeed, if $h < 0$, then $|h| = -h$ and so $\frac{|h|}{h} = -1$. If $h > 0$, $\frac{|h|}{h} = 1$. Thus
$$\lim_{h \to 0^-} \frac{|h|}{h} = -1, \qquad \lim_{h \to 0^+} \frac{|h|}{h} = 1.$$
Since the one-sided limits are different, the limit does not exist. Thus $f'(\frac{1}{2})$ does not exist.

Note that
$$f(-x) = |-x - \frac{1}{2}||-x + \frac{1}{2}| = |-(x + \frac{1}{2})||-(x - \frac{1}{2})| = f(x).$$
Moreover,
$$\frac{df(-x)}{dx} = -f'(x),$$
by the chain rule. Thus if $f'(x)$ exists, then so must $f'(-x)$. We conclude that since $f'(\frac{1}{2})$ does not exist, then $f'(-\frac{1}{2})$ cannot exist, either. Thus $f(x)$ is not differentiable at either of the two points $\pm\frac{1}{2}$. □

Problem 8. A function $f(x)$ and its first two derivatives are graphed below. Label which one is which.



Solution. Let us label the graphs as in the picture and let us call the function $f(x)$.

Suppose that the graph of $f(x)$ is given by (c). Since $f(x)$ is increasing, its derivative is positive, and so its graph could be either (a) or (b). Suppose that (a) is the graph of $f'(x)$, so that the graph of $f''(x)$ is the remaining choice, (b). Since the function with graph (a) is decreasing, its derivative $f''(x)$ must be negative. But this contradicts the assumption that the graph of $f''(x)$ is (b). Similarly, if we assume that the graph of $f'(x)$ is (b), so that the graph of $f''(x)$ is (a), we again arrive at a contradiction, since $f'(x)$ is decresing, so that $f''(x) < 0$, which is not the case with (a).

Suppose next that the graph of $f(x)$ is given by (a). Since $f(x)$ is decreasing, its derivative must be negative, so that the graph of $f'(x)$ is (c). The slope of the graph (a) at $x \approx 0.4$ appears to be -1, since the line with this slope is tangent to (a). Thus $f'(0.4) \approx -1$. However, we see that (c) is approximately -0.4 at $x \approx 0.4$, which is a contradiction.

Suppose lastly that the graph of $f(x)$ is given by (b). Since $f(x)$ is decreasing, its derivative must be (c) and its second derivative must be (a). This is the only remaining choice and must therefore be the answer. $\square$

Sample Midterm Math 31A

Student ID : ____________________

First Name: ____________________

Last Name: ____________________

There are a total of 5 problems. SHOW YOUR WORK ON ALL PROBLEMS. Please write clearly.

1. A conical tank has height 3 m and radius 2 m at the top. Water flows in at a rate of 2 m^3/min. How fast is the water level rising when it is 2 m?

2. Find the points on the graph of $x^3 - y^3 = 3xy - 3$ where the tangent line is horizontal.

3. Let f be the function defined by $f(x) = |x^2 - 1|$. Find the points c (if any) such that $f'(c)$ does not exist.

4. (i) Find numbers a and b such that $\lim_{x\to 0} \frac{\sqrt{ax+b}-2}{x} = 1$.

   (ii) Evaluate the limit $\lim_{x\to 0} x[\frac{1}{x}]$, where $[x]$ denotes the greatest integer function.

5. Suppose that f and g are differentiable functions such that $f(g(x)) = x$ and $f'(x) = 1 + [f(x)]^2$. Show that $g'(x) = \frac{1}{1+x^2}$.

1. Differentiate

(7 points) (a) $$f(x) = \frac{\sqrt{x}}{1+x^2}$$

(13 points) (b) $$f(x) = \csc^2(\cos x)$$

(a) $$f'(x) = \frac{\frac{1}{2}x^{-1/2}(1+x^2) - x^{1/2}(2x)}{(1+x^2)^2}$$

(b) $$f'(x) = 2\csc(\cos x)(-\csc(\cos x)\cot(\cos x))(-\sin x)$$

2. Suppose $f(x)$ is a one-to-one function with the property

$$f'(x) = 1 + (f(x))^2$$

for all x. Let $g(x) = f^{-1}(x)$ be the inverse of $f(x)$. Show that

$$g'(x) = \frac{1}{1 + x^2}.$$

$$g'(x) = \frac{1}{f'(g(x))}$$

$$f'(g(x)) = 1 + (f(g(x)))^2 = 1 + x^2$$

$$g'(x) = \frac{1}{1+x^2}$$

3. There are two values of x for which the tangent to the curve

$$x^2 - xy + 2y^2 = 3$$

at (x, y) is horizontal. Find those values of x.

$$2x - y - x\frac{dy}{dx} + 4y\frac{dy}{dx} = 0$$

$$(4y - x)\frac{dy}{dx} = y - 2x$$

$$\frac{dy}{dx} = \frac{y - 2x}{4y - x} = 0 \quad \text{if} \quad y = 2x$$

$$x^2 - x(2x) + 2(2x)^2 = 3$$

$$7x^2 = 3$$

$$x = \pm\sqrt{\frac{3}{7}}$$

4. Use the **definition** of the derivative to show that if

$$g(x) = xf(x)$$

then

$$g'(x) = f(x) + xf'(x).$$

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)f(x+h) - xf(x)}{h}$$

$$= \lim_{h \to 0} \frac{xf(x+h) + hf(x+h) - xf(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}f(x+h)}{\cancel{h}} + x\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} f(x+h) + \lim_{h \to 0} x \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= f(x) + xf'(x)$$

5. There is a number a for which

$$\lim_{x \to 2} \frac{ax^3 - 7x^2 + ax - 2}{x - 2}$$

exists. Find the value of a and then evaluate the limit.

$$a(2)^3 - 7(2)^2 + a(2) - 2 = 0$$

$$10 \quad -30 = 0 \qquad a = 3$$

$$\begin{array}{r|l} & 3x^2 - x + 1 \\ \hline x - 2 & 3x^3 - 7x^2 + 3x - 2 \\ & \underline{3x^3 - 6x^2} \\ & \quad -x^2 + 3x \\ & \quad \underline{-x^2 + 2x} \\ & \qquad x - 2 \end{array}$$

$$\lim_{x \to 2} \frac{3x^3 - 7x^2 + 3x - 2}{x - 2}$$

$$= \lim_{x \to 2} \frac{\cancel{(x-2)}(3x^2 - x + 1)}{\cancel{x-2}}$$

$$= 3(2)^2 - (2) + 1 = 11$$

(15) 1. Evaluate the following limits:

$$\lim_{h\to 0}\frac{(1+h)^3-1}{h}=\lim_{h\to 0}\frac{3h+3h^2+h^3}{h}=\lim_{h\to 0}(3+3h+h^2)=3.$$

$$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}=\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}\Big(\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\Big)$$

$$=\lim_{x\to 0}\frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})}=1.$$

(10) 2. Determine the point(s) at which the function

$$f(u)=\frac{u+1}{u^2-9}$$

is discontinuous. For each such point a, find $\lim_{u\to a+} f(u)$.

Solution: The function is continuous at every point, except possibly points at which the denominator is 0, which are $a=\pm 3$. Near each of these points, the denominator is close to 0, but the numerator is not. Therefore, the limits are $\pm\infty$. To decide the correct sign, look at the signs of the numerator and denominator for u close to a:

$$\lim_{u\to -3+} f(u)=+\infty, \quad \lim_{u\to 3+} f(u)=+\infty.$$

(14) 3. Compute the following derivatives:

$$(a)\ \frac{d}{dx}(x^2\cos x)=2x\cos x-x^2\sin x$$

$$(b)\ \frac{d}{dt}\Big(\frac{3t^2+1}{5t^3+7t}\Big)=\frac{(5t^3+7t)(6t)-(3t^2+1)(15t^2+7)}{(5t^3+7t)^2}=\frac{-15t^4+6t^2-7}{t^2(5t^2+7)^2}$$

(11) 4. Let $f(x)=3x^2+1$ and $g(x)=2x^3-12x$. Find the values of x for which the tangent line to the graph of f at $(x,f(x))$ and the tangent line to the graph of g at $(x,g(x))$ are parallel.

Solution: Need to find those x's for which $f'(x)=g'(x)$, since this means the two tangent lines have the same slope. So, need to solve $6x=6x^2-12$ for x. The solutions are -1 and 2.

(11) 5. The position of a car at time t is given by

$$s(t)=t^3-6t^2+15t+1.$$

Find all times t at which the car is accelerating.

Solution: The acceleration at time t is $s''(t) = 6t - 12$. This is positive if and only if $t > 2$.

(14) 6. Differentiate the following functions:

$$\frac{d}{dx}\sqrt{4 - 3\cos x} = \frac{1}{2}(4 - 3\cos x)^{-1/2}\frac{d}{dx}(4 - 3\cos x) = \frac{3\sin x}{2\sqrt{4 - 3\cos x}}.$$

$$\frac{d}{dx}\tan(x^2) = 2x\sec^2(x^2).$$

(10) 7. Find the equation for the tangent line to the curve

$$x^2y^3 + 2y = 3x$$

at the point $(2, 1)$.

Solution: Differentiate implicitly with respect to x to get

$$2xy^3 + 3x^2y^2y' + 2y' = 3.$$

When $x = 2, y = 1$, this gives $y' = -1/14$. So, the equation for the tangent line is $y - 1 = -(1/14)(x - 2)$, or

$$y = -\frac{1}{14}x + \frac{8}{7}.$$

(15) 8. Prove the following:

$$\lim_{h \to 0} \frac{\sin h}{h} = 1.$$

See page 80 of the text.