Course Announcement

Surreal Numbers

Math 285D, Fall Quarter 2014

MWF 11–11:50AM, MS 5138

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Office hours. M 1 pm–1:50 pm; MS 5614.

Description. The surreal numbers were discovered by John H. Conway while studying combinatorial games, and popularized by Donald E. Knuth, who coined the term "surreal number." They were further studied by Alling, Gonshor, van den Dries-Ehrlich, Kruskal, Lurie, and others.

The surreal numbers form a proper class **No** which contains the set of real numbers and the class of ordinals, and more. They come equipped with a natural linear ordering as well as arithmetic operations, turning **No** into a a real closed ordered field, as Conway showed. So for example, $\omega - \pi$, $-1/\omega$, $\sqrt{\omega}$, etc. (where ω is the first transfinite ordinal) make sense as surreal numbers. Kruskal defined an exponential function on **No**, and van den Dries and Ehrlich proved that as an exponential field, **No** is an elementary extension of the real exponential field $(\mathbb{R}, +, \times, \exp)$ (and hence, by a famous theorem of Wilkie, model complete and o-minimal). This result remains true if in addition **No** is equipped with structure coming from certain analytic functions on bounded domains.

The goal of this course is to present the construction and basic facts about **No** and to prove the theorem of van den Dries and Ehrlich. Conway originally constructed **No** by a combination of von Neumann's construction of ordinal numbers and the Dedekind construction of the completion of an ordered set. I will follow an alternative treatment by Gonshor.

The class **No** has a very rich structure, much of which still remains in the dark (e.g., its connection to asymptotics and transseries). Open problems will be pointed out along the way. Prerequisites are some basic knowledge of naive set theory (e.g., ordinals), first-order logic, and algebra. If in doubt, ask me.

References.

Norman L. Alling, *Foundations of Analysis over Surreal Number Fields,* North-Holland Mathematics Studies, vol. 141, North-Holland Publishing Co., Amsterdam, 1987.

John H. Conway, *On Numbers and Games,* London Mathematical Society Monographs, no. 6, Academic Press, London-New York, 1976.

Jan Denef, Lou van den Dries, *p-adic and real subanalytic sets,* Ann. of Math. (2) **128** (1988), no. 1, 79–138.

Lou van den Dries, Philip Ehrlich, *Fields of surreal numbers and exponentiation,* Fund. Math. **167** (2001), no. 2, 173–188.

Lou van den Dries, Angus Macintyre, Dave Marker, *The elementary theory of restricted analytic fields with exponentiation,* Ann. of Math. (2) **140** (1994), no. 1, 183205.

Harry Gonshor, *An Introduction to the Theory of Surreal Numbers,* London Mathematical Society Lecture Note Series, vol. 110, Cambridge University Press, Cambridge, 1986.

Donald E. Knuth, *Surreal Numbers,* Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1974.