Problem Set 7
Due Friday, May 30.

Mathematical Logic

Math 114L, Spring Quarter 2008

1. (20 pt.) Let $\mathfrak{A}$ be a structure. Show:
   (a) If α and β are automorphisms of $\mathfrak{A}$, then so is $\alpha \circ \beta$.
   (b) If α is an automorphism of $\mathfrak{A}$, then so is α^{-1}.

   (For those of you who know about groups: this yields that the set of all automorphisms of $\mathfrak{A}$ forms a group, with $\circ$ as group operation, called the **automorphism group** of $\mathfrak{A}$.)
2. (20 pt.) Consider the first-order language whose only parameter is a 2-place relation symbol $<$. Without proof: what are all automorphisms of the structure $\mathfrak{Z} = (\mathbb{Z}, <^{\mathfrak{Z}})$, where $<^{\mathfrak{Z}}$ is the usual ordering on the set $\mathbb{Z}$ of integers?
3. (20+10 pt.) Let $\mathfrak{A}$ be an structure, and suppose that D and E are subsets of A^k which are definable in $\mathfrak{A}$.
   (a) Show that $\emptyset$, $D \cap E$, $D \cup E$, and $A^k \setminus D$ are definable in $\mathfrak{A}$.
   (b) Show that if $k > 1$, then $\pi(D) \subseteq A^{k-1}$ is definable in $\mathfrak{A}$; here $\pi \colon A^k \to A^{k-1}$ is given by $\pi(a_1, \dots, a_k) = (a_1, \dots, a_{k-1})$.
4. (10+20 pt.) A wff which does not contain $\neg$ is called **positive.**
   (a) Give an inductive definition of the set of positive wffs (similar to the definition of the set of all wffs).
   (b) Show that every positive formula is satisfiable.
5. (20 pt. extra credit.) Consider the first-order language with a single 1-place function symbol f. Find a sentence φ such that every structure satisfying φ has infinite universe. Also, do the same problem for the language with a single is a 2-place relation symbol R.