Problem Set 5
Solutions

Mathematical Logic

Math 114L, Spring Quarter 2008

1. Very similar to the proof that for any term t we have $K(t) = 1$, and if t' is a proper initial segment of a term, then $K(t') < 1$. (More details shall be given in the discussion section.)

2. Here the structure, call it $\mathfrak{N}$, is $(\mathbb{N}; +, \cdot)$.

   (a) $\{0\}$ is defined in $\mathfrak{N}$ by $\boldsymbol{v}_1 + \boldsymbol{v}_1 = \boldsymbol{v}_1$. Also by $\forall\, \boldsymbol{v}_2(\boldsymbol{v}_2 + \boldsymbol{v}_1 = \boldsymbol{v}_2)$. Also by $\forall\, \boldsymbol{v}_2(\boldsymbol{v}_2 \cdot \boldsymbol{v}_1 = \boldsymbol{v}_1)$.

   (b) $\{1\}$ is defined in $\mathfrak{N}$ by $\boldsymbol{v}_1 + \boldsymbol{v}_1 \neq \boldsymbol{v}_1 \wedge \boldsymbol{v}_1 \cdot \boldsymbol{v}_1 = \boldsymbol{v}_1$. Also by $\forall\, \boldsymbol{v}_2(\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = \boldsymbol{v}_2)$.

   (c) For the successor relation, two defining formulas are

   $$\forall\, \boldsymbol{v}_3(\theta(\boldsymbol{v}_3) \rightarrow \boldsymbol{v}_2 = \boldsymbol{v}_1 + \boldsymbol{v}_3) \quad \text{and} \quad \exists\, \boldsymbol{v}_3(\theta(\boldsymbol{v}_3) \wedge \boldsymbol{v}_2 = \boldsymbol{v}_1 + \boldsymbol{v}_3)$$

   where $\theta(\boldsymbol{v}_1)$ defines $\{1\}$. Also, successor has the quantifier-free definition:

   $$\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = \boldsymbol{v}_1 \cdot \boldsymbol{v}_1 + \boldsymbol{v}_1 \wedge (\boldsymbol{v}_1 + \boldsymbol{v}_1 = \boldsymbol{v}_1 \rightarrow \boldsymbol{v}_2 \cdot \boldsymbol{v}_2 = \boldsymbol{v}_2 \wedge \boldsymbol{v}_1 \neq \boldsymbol{v}_2)$$

   (that is, $xy = x^2 + x \wedge (x = 0 \rightarrow y = 1)$).

   (d) Ordering $<$ is defined by $\exists\, \boldsymbol{v}_3(\boldsymbol{v}_2 = \boldsymbol{v}_1 + \boldsymbol{v}_3 \wedge \boldsymbol{v}_1 \neq \boldsymbol{v}_2)$. (Cf. page 91.)

3. (a) The formula $\exists z\ z \cdot z = x$ defines $[0, \infty)$ in $\mathfrak{R}$.

   (b) The formula $\exists z(\forall y(z \cdot y = y) \wedge x = z + z)$ defines $\{2\}$ in $\mathfrak{R}$.

4. The sentence $\forall\, x\ \exists\, y\, Pyx$ is true in $(\mathbb{R}; <)$ and false in $(\mathbb{N}; <)$.

5. (a) Consider the first-order language with equality but without any other function, predicate or constant symbols. For $k \in \mathbb{N}$, $k \geq 1$ consider the following sentence in this language:

   $$\varphi_k = \exists y_1 \cdots \exists y_k \forall x \left(\left(\bigwedge_{1 \leq i < j \leq k} \neg y_i = y_j \right) \wedge \left(\bigvee_{1 \leq i \leq k} x = y_i \right) \right).$$

(Here and below we use the useful abbreviation

$$\bigwedge_{i \in I} \varphi_i := \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_n}$$

for a finite index set $I = \{i_1, \ldots, i_n\}$ and formulas $\varphi_{i_1}, \ldots, \varphi_{i_n}$, and similarly for $\bigvee$.) Then a structure $\mathfrak{A}$ satisfies φ if and only if the universe A of $\mathfrak{A}$ has exactly k elements. Hence given a finite subset A of $\mathbb{N}^{>0}$, the sentence $\varphi_A = \bigvee_{k \in A} \varphi_k$ shows that A is a spectrum.

(b) Consider the first-order language with equality and a 2-place predicate symbol E, and let ψ be the conjunction of the three axioms for equivalence relations:

(E1) $\forall x Exx$;

(E2) $\forall x \forall y (Exy \leftrightarrow Eyx)$;

(E3) $\forall x \forall y \forall z (Exy \wedge Eyz \to Exz)$.

For $m \geq 1$ let ψ_m be the sentence

$$\forall x \exists y_1 \cdots \exists y_m \left(\bigwedge_{1 \leq i < j \leq m} \neg y_i = y_j \wedge \forall z \left(Exz \to \bigvee_{1 \leq i \leq m} Ey_i z \right) \right).$$

Then a structure $\mathfrak{M} = (M, E^{\mathfrak{M}})$ satisfies $\varphi_m := \psi \wedge \psi_m$ if and only if $E^{\mathfrak{M}}$ is an equivalence relation on M all of whose equivalence classes have exactly m elements. Hence a finite structure satisfying φ_m has km elements, for some $k \geq 1$. Conversely, for every natural number of the form km ($k \in \mathbb{N}$, $k \geq 1$) it is easy to find an equivalence relation $E^{\mathfrak{M}}$ on $M := \{1, \ldots, km\}$ with k equivalence classes of m elements each: $(a, b) \in E^{\mathfrak{M}} \iff m$ divides $a - b$ (in $\mathbb{Z}$). Then $\mathfrak{M} = (M, E^{\mathfrak{M}})$ satisfies φ_m. This shows that the set

$$A = \left\{ km : k \in \mathbb{N}^{>0} \right\}$$

of multiples of m is a spectrum.

(c) Let A and B be spectra, given by sentences σ and τ in certain first-order languages. Then $A \cap B$ and $A \cup B$ are given by $\sigma \wedge \tau$ and $\sigma \vee \tau$, respectively, where we consider $\sigma \wedge \tau$ and $\sigma \vee \tau$ as sentences in the disjoint union of the two first-order languages.