1. Prove *Hurwitz Theorem:* Suppose $\{f_n\}$ and f are holomorphic functions on a connected open set $\Omega \subset \mathbb{C}$. If $f_n \to f$ uniformly on compact sets and $f \not\equiv 0$, then the zeros of f_n converge to those of f in the following sense: given a ball B with $\bar{B} \subseteq \Omega$ and f nowhere zero on ∂B, the number of zeros (counting multiplicity) of f_n in B converges to the number for f. [*Remark:* Since the number is an integer, convergence means eventual equality.]

2. Show that for each integer $m \geq 1$, the following two sets of mappings have precisely the same elements:
(i) The m-fold holomorphic branched covering maps $f : \mathbb{D} \to \mathbb{D}$. That is, holomorphic functions f such that for each $w \in \mathbb{D}$ there are exactly m solutions (counting multiplicity) to $f(z) = w$ with $z \in \mathbb{D}$.
(ii) Blaschke products of degree m, that is, functions of the form

$$f(z) = e^{i\phi} \prod_{j=1}^{m} \frac{z - z_j}{1 - \bar{z}_j z}$$

with $\phi \in [0, 2\pi)$ and $z_1, \ldots, z_m \in \mathbb{D}$.
[*Remark:* The case $m = 1$ is the classification of disk automorphisms, which was done in class.]

3. Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic with $f(0) = 0$ and $f'(0) = 1$ and let

$$M := \sup_{z \in \mathbb{D}} |f(z)|$$

(a) Explain why $M \geq 1$.
(b) Show that

$$|f(z) - z| \leq \frac{1}{12M} \qquad \text{for} \quad |z| = \frac{1}{4M},$$

for example, by estimating the power series.
(c) Deduce that $f(\mathbb{D})$ contains $\{z : |z| < \frac{1}{6M}\}$.

4. Let $\Omega \subseteq \mathbb{C}$ be *convex* and open. Use the sketch below to prove the following: If $f : \mathbb{D} \to \Omega$ is a biholomorphism (onto Ω), then $G_r := f(\{|z| \leq r\})$ is convex for each $0 \leq r < 1$.
(a) Argue that it suffices to show that for all $|z_1| \leq |z_2| < 1$ the line segment joining $f(z_1)$ to $f(z_2)$ is inside the image of the ball $\{z : |z| \leq |z_2|\}$.
(b) For $t \in [0, 1]$ fixed, apply Schwarz Lemma to

$$z \mapsto f^{-1}\big(tf(zz_1/z_2) + (1 - t)f(z)\big).$$

(c) Now imagine that Ω is *star-shaped* with respect to the origin and $f : \mathbb{D} \to \Omega$ is a biholomorphism (onto Ω) with $f(0) = 0$. Show that each set $G_r := f(\{|z| \leq r\})$ is star-shaped with respect to the origin.

Remark: These are theorems of E. Study, while the proof is that of T. Radó. In the next homework, you will prove the Riemann mapping theorem which guarantees the existence of such a biholomorphism (except in the case $\Omega = \mathbb{C}$, when it is not possible).

5. Use the Poisson integral formula to prove the following version of Harnack's inequality for the unit ball $B(0,1) \subseteq \mathbb{R}^n$ (and $n \geq 2$): If $u : B(0,1) \to [0,\infty)$ is harmonic, then

$$\frac{1-|x|}{(1+|x|)^{n-1}}u(0) \leq u(x) \leq \frac{1+|x|}{(1-|x|)^{n-1}}u(0)$$

for each $|x| < 1$. Moreover, equality can occur, so these estimates are best possible.

6. Give a direct proof of Montel's Theorem (families uniformly bounded on compacta are normal) from the Cauchy Integral and Arzelà–Ascoli Theorems. (The in-class proof is via the analogue for harmonic functions.)

7. Suppose $\Omega \subset \mathbb{C}$ is open and connected.
(a) Let $f_n : \Omega \to \mathbb{C}$ be univalent (=holomorphic and injective) and converge uniformly on compact sets to some (holomorphic) $f : \Omega \to \mathbb{C}$. Show that f is univalent or constant.
(b) Suppose further that Ω has compact closure and $z_0 \in \Omega$. Show that among all univalent maps $f : \Omega \to \mathbb{D}$ that obey $f(z_0) = 0$, there is (at least) one that achieves the maximal value of $\operatorname{Re} f'(z_0)$. [*Note:* you will need to verify that the set of maps is non-empty and that $\operatorname{Re} f'(z_0)$ is bounded.]
(c) Explain why any such maximal f has $f'(z_0) > 0$.