1. Let $\Omega \subseteq \mathbb{C}$ be open, connected, and simply connected. Let $f : \Omega \to \mathbb{C} \setminus \{0\}$ be holomorphic. Show the following:
(a) There is a holomorphic function $g : \Omega \to \mathbb{C}$ so that $f(z) = e^{g(z)}$; moreover if $\tilde{g} : \Omega \to \mathbb{C}$ is holomorphic and $f(z) = e^{\tilde{g}(z)}$, then $g(z) - \tilde{g}(z)$ is constant and equal to some element of $2\pi i\mathbb{Z}$.
(b) There is a holomorphic function $h : \Omega \to \mathbb{C}$ so that $f(z) = [h(z)]^2$. Except $\tilde{h}(z) = -h(z)$ there are no other holomorphic functions with this property.

2. Fix $0 < r < R < \infty$ and let $A := \{z \in \mathbb{C} : r \leq |z| \leq R\}$. Suppose $f : A \to \mathbb{C}$ is continuous and is holomorphic on the interior of A, which we denote A°.
(a) Express $f(z)$ for each $z \in A^\circ$ as a Cauchy-like integral over the two circles $|z| = r$ and $|z| = R$.
(b) Deduce that there is a holomorphic function g on $\{z : |z| > r\}$ and another holomorphic function h on $\{z : |z| < R\}$ so that $f(z) = g(z) + h(z)$ for all $z \in A^\circ$.
(c) Conclude that there are coefficients c_n, $n \in \mathbb{Z}$, so that

$$f(z) = \sum_{-\infty}^{\infty} c_n z^n$$

as a uniformly convergent sum on all compact subsets of A°.

Remark: The series above is called a Laurent series for f. Suppose now that f is holomorphic in the 'annulus' $\{z \in \mathbb{C} : 0 < |z| < R'\}$. Choosing some $0 < r < R < R'$ we can apply the analysis above to get a Laurent series representation. Close inspection of your argument should reveal that the coefficients c_n do not depend on the particular choice of r or R. Thus, we have a Laurent expansion valid in throughout $\{z \in \mathbb{C} : 0 < |z| < R'\}$.

3. Let $E \subseteq [0, 1] \subseteq \mathbb{R} \subseteq \mathbb{C}$ denote the usual Cantor ternary set and let $B = \{z \in \mathbb{C} : |z| < 2\}$. Suppose $f : B \setminus E \to \mathbb{C}$ is bounded and holomorphic. Show that f admits a holomorphic extension to all of B.

4. (a) Evaluate

$$\int_0^\infty \frac{\sqrt{x}}{x^2 + 1}\, dx$$

by writing $2i\sqrt{x} = \lim_{\varepsilon \to 0} \exp\{\frac{1}{2}\text{Log}(-x + i\varepsilon)\} - \exp\{\frac{1}{2}\text{Log}(-x + i\varepsilon)\}$.
(b) Evaluate

$$\int_0^{2\pi} [\sin\theta]^{2k}\, d\theta$$

for all integers $k \geq 1$. [*Hint:* What is $(z - z^{-1})$ when $z = e^{i\theta}$.]

5. In class, we saw that

$$\zeta(s) = \frac{1}{2i\cos(\pi s/2)} \int_\gamma \frac{z^{-s}\, dz}{e^{-2\pi z} - 1}.$$

where $\gamma : \mathbb{R} \to \mathbb{C}$ by $\gamma(t) = \frac{1}{2} - |t| + it$. Initially this was for $\operatorname{Re} s > 1$ and $s \notin 2\mathbb{Z}+1$ for other values of s the integral can be regarded as the definition of $\zeta(s)$ (as a meromorphic function). For this problem, we consider only $s \in \mathbb{C}$ with $\operatorname{Re} s < 0$.
(a) By collapsing the contour onto $(-\infty, 0]$ evaluate

$$\int_\gamma e^{2\pi n z} z^{-s}\, dz$$

for each integer $n \geq 1$. (Note that the answer involves the Γ function.)
(b) Similarly show that

$$\int_\gamma \frac{e^{2\pi N z}}{e^{-2\pi z} - 1} z^{-s}\, dz$$

converges to 0 as $N \to \infty$.
(c) By writing

$$\frac{1}{e^{-2\pi z} - 1} = \frac{e^{2\pi N z}}{e^{-2\pi z} - 1} + \sum_{n=1}^{N} e^{2\pi n z}$$

deduce that ζ obeys the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\tfrac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$$

and so conclude (using results from class and previous homework) that $s = 1$ is the only (non-removable) singularity of ζ.

6. Evaluate

$$\int_{\mathbb{R}} e^{-t^2/2} e^{i\xi t}\, dt$$

for all $\xi \in \mathbb{C}$. Deduce *Wick's Theorem* (as it is called in the physics literature):

$$\tfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} t^{2k} e^{-t^2/2}\, dt = (2k-1)!! = (2k-1)(2k-3)\cdots(3)(1).$$

Convince yourself that this is the number of ways of pairing off $2k$ objects.