246A Homework 3 Due Oct. 26.

1. Let $P \in \mathbb{R}[x]$ be monic and non-constant (that is, a non-constant polynomial in with real coefficients whose leading coefficient is unity). Show that P can be written uniquely as a product of linear and irreducible quadratic monic polynomials with real coefficients.

2. Let $\mathcal{A}$ be an associative division algebra (containing the identity Id) consisting of $n \times n$ real matrices (with usual matrix addition/multiplication).
(a) Show that the minimal polynomial (over $\mathbb{R}$) of every non-zero $A \in \mathcal{A}$ has degree one or two; moreover, if the degree is not two, show that A is a scalar (i.e. $\mathbb{R}$) multiple of the identity.
(b) Suppose $A \in \mathcal{A}$ is not a scalar multiple of the identity, then (i) $\operatorname{span}_{\mathbb{R}}\{\mathrm{Id}, A\}$ contains a square-root of $-\mathrm{Id}$ (indeed, it contains a solution to every quadratic $P \in \mathbb{R}[x]$) and (ii) if $B \in \mathcal{A}$ and $AB = BA$ then $B \in \operatorname{span}_{\mathbb{R}}\{\mathrm{Id}, A\}$.
(c) Show that if $\dim_{\mathbb{R}}(\mathcal{A}) \geq 3$ then there are $i, j \in \mathcal{A}$ so that

$$i^2 = j^2 = -\mathrm{Id} \quad \text{and} \quad ij = -ji \neq 0.$$

[*Hint:* if $i, \ell \in \mathcal{A}$ and $i^2 = \ell^2 = -\mathrm{Id}$, then $i\ell + \ell i$ commutes with i and ℓ. Now try $j = \alpha\ell + \beta i$.]
(d) Further show that $k := ij \notin \operatorname{span}_{\mathbb{R}}(\mathrm{Id}, i, j)$ and thus $\dim_{\mathbb{R}}(\mathcal{A}) \geq 4$. [*Hint:* $i(ij)i = j(ij)j = ij$.] Moreover $k^2 = -\mathrm{Id}$ and $ijk = -\mathrm{Id}$.

Remarks: By the regular representation introduced in class, any abstract finite dimensional associative algebra over $\mathbb{R}$ can be realized as a concrete algebra of matrices. Matrix algebras introduced in this way have n equal to the dimension of $\mathcal{A}$ as a vector space (over $\mathbb{R}$), but the example of $\mathcal{A}$ being all real multiples of Id shows that n can be much larger. What we see now is that an algebra of matrices of this sort must 'behave like' $\mathbb{R}$ if it is one dimensional, like $\mathbb{C}$ if two dimensional, and must 'contain' a copy of the quaternions if the dimension is three or higher. To finish the proof of the Frobenius theorem, we need to show that that in the last case, $\mathcal{A} = \operatorname{span}_{\mathbb{R}}\{\mathrm{Id}, i, j, k\}$.

Suppose $\mathcal{A}$ has five or more dimensions, then by (b) there is an element $\ell \notin \operatorname{span}\{\mathrm{Id}, i, j, k\}$ with $\ell^2 = -\mathrm{Id}$. Moreover, as argued for (c), we have that $i\ell + \ell i = \gamma_1 \mathrm{Id}$, $j\ell + \ell j = \gamma_2 \mathrm{Id}$, and $k\ell + \ell k = \gamma_3 \mathrm{Id}$ for some real constants $\gamma_1, \gamma_2, \gamma_3$. A little messing around then reveals $m := 2\ell + \gamma_1 i + \gamma_2 j + \gamma_3 k$ obeys

$$im + mi = jm + mj = km + mk = 0 \quad \text{and hence} \quad m = ijkmkji = -m.$$

Thus $m = 0$, contradicting that $\ell \notin \operatorname{span}\{\mathrm{Id}, i, j, k\}$.

3. Let $\Omega \subseteq \mathbb{C}$ be open and connected. Suppose f and g are meromorphic on Ω and $g \not\equiv 0$. Show that f/g is meromorphic (after removing removable singularities).

Remarks: With a little extra work, this shows that the set of meromorphic functions on Ω is an infinite-dimensional (commutative) division algebra (over $\mathbb{R}$ and $\mathbb{C}$).

4. Prove the *Casorati–Weierstrass Theorem:* Suppose f is defined in a deleted neighbourhood of $z_0 \in \mathbb{C}$, say $\{z : 0 < |z - z_0| < r\}$, and is holomorphic there. Suppose also that z_0 is neither a removable singularity nor a pole. Prove that for each $0 < \epsilon < r$

the set $f(\{z : 0 < |z - z_0| < \epsilon\})$ is dense in $\mathbb{C}$.

5. Let us say that meromorphic function f on $\mathbb{C}$ is *doubly periodic* if there exists $\tau \in \mathbb{C}$ with $\operatorname{Im} \tau > 0$ and

$$f(z + \tau) = f(z + 1) = f(z).$$

Notice that we have fixed one of the periods to be 1. This is no real loss of generality since other cases can be reduced to this via a change of variable $z \mapsto az$. The number τ is far from unique as we will discuss later in the course. For now, show that
(a) Holomorphic doubly periodic functions are constant. Indeed,
(b) Nonconstant doubly periodic functions are onto $\mathbb{C} \cup \{\infty\}$, that is, they have a pole and achieve every (finite) value in $\mathbb{C}$.
Remark: It has been reported (though documentary evidence is limited) that Liouville lectured on the result (a), Cauchy objected that this would imply that bounded holomorphic functions are constant, and so what we now call Liouville's Theorem was born. Note that obvious candidate for Cauchy's argument uses the fact that there is a doubly periodic function with property (b), which was known by direct construction (we'll do this later too).