246A Homework 2 Due Oct. 19.

1. (a) Let $\Omega \subseteq \mathbb{C}$ be open. Suppose $f : \Omega \to \mathbb{C}$ is holomorphic; show

$$\Delta f = 4\partial_z \partial_{\bar{z}} f = 0,$$

where Δ is the Laplacian.
(b) Let $f : \mathbb{C} \to \mathbb{C}$ be continuous and a distributional solution of $\partial_{\bar{z}} f = 0$. Show that f is distributionally harmonic:

$$\int_{\mathbb{C}} f(x+iy)[\Delta\phi](x+iy)\,dx\,dy = 0 \quad \text{for all} \quad \phi \in C_c^\infty(\mathbb{C}).$$

(c) Using just the one-variable Fundamental Theorem of Calculus (and the equality of double Riemann integrals with iterated integrals), prove Green's theorem in the following form:

$$2i \iint_T [\partial_{\bar{z}} f](x+iy)\,dx\,dy = \int_{\partial T} f(z)\,dz$$

where $f : \mathbb{C} \to \mathbb{C}$ is C^1 and T is the triangle with vertices $0 + i0$, $a + i0$, and $0 + ib$, where $a, b > 0$.
Remark: The parts are related by the appearance of $\partial_{\bar{z}}$; they don't build on one another. Be explicit about how you parameterize the curve ∂T. Weyl's lemma (which we will prove in due course) says that distributionally harmonic functions are C^∞. It then follows that continuous functions that are distribution solutions of $\partial_{\bar{z}} f = 0$ are actually holomorphic in the usual sense.

2. Let $\Omega \subseteq \mathbb{C}$ be open and let $f_n : \Omega \to \mathbb{C}$ be holomorphic. Suppose that for each compact set $K \subset \Omega$ the functions f_n converge uniformly to some $f : \Omega \to \mathbb{C}$. Show that f must me holomorphic. Further, show that $f_n'(z) \to f'(z)$ uniformly on compact subsets of Ω.

3. (a) Prove *Liouville's Theorem*: Suppose $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and

$$|f(z)| \leq C(1 + |z|)^n$$

for some $C > 0$ and integer $n \geq 0$, then f is a polynomial of degree not exceeding n.
(b) Let Ω be an open neighbourhood of $0 \in \mathbb{C}$. Suppose g is holomorphic on $\Omega \setminus \{0\}$ and obeys

$$|g(z)| \leq C|z|^{-n}$$

there (with n and C as before). Show that there is a holomorphic function $h : \Omega \to \mathbb{C}$ and coefficients $a_1, \ldots, a_n$ in $\mathbb{C}$ so that

$$g(z) = h(z) + \sum_{k=1}^{n} a_k z^{-k}.$$

The sum here is called the *principal part* of g at the point 0.
Hint: First treat the case $n = 0$, for which the sum is empty (and so zero). This $n = 0$ result is known as *Riemann's removable singularity theorem.* For general n consider $f(z) = z^n g(z)$.

4. For $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$, let us define

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt$$

(as an improper Riemann integral).
(a) Prove that Γ is holomorphic on this region.
(b) Show that $z\Gamma(z) = \Gamma(z+1)$ when $\mathrm{Re}(z) > 0$.
(c) Deduce that $\Gamma(n+1) = n!$ when $n \geq 0$ is an integer.
(d) Argue that there is a (unique) extension of Γ to a holomorphic function on $\mathbb{C} \setminus \{0, -1, -2, \dots\}$ that obeys $z\Gamma(z) = \Gamma(z+1)$. Show that the omitted points are polar singularities and determine their principal part.

5. (a) Show that

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

defines a holomorphic function on $\{s \in \mathbb{C} : \mathrm{Re}(s) > 1\}$.
(b) Show that

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

You may take the fundamental theorem of arithmetic for granted, but you must address the issue of convergence.
(c) Identify a function $g : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ so that

$$f(n) = \int_{-1/2}^{1/2} f(n+x)\, dx + \int_{-1/2}^{1/2} f''(n+x) g(x)\, dx$$

for all C^∞ functions f defined in a neighbourhood of $[n - \frac{1}{2}, n + \frac{1}{2}]$.
[*Remark:* one may view this formula as giving the error made when using the midpoint rule of numerical integration.]
(d) Use part (c) to show that we can extend the definition of ζ to make it meromorphic in the region $\mathrm{Re}(s) > -1$. Identify the (single) pole and its residue.