1. Let A be an $N \times N$ matrix of real numbers. Show that

$$e^A = \lim_{n\to\infty} \left(1 + \tfrac{1}{n}A\right)^n$$

[*Hint:* Prove $\|e^{A/n} - (1 + \frac{1}{n}A)\| = O(n^{-2})$ as $n \to \infty$.]

2. Let $\gamma : [0,1] \to \mathbb{C}$ be a rectifiable curve. Show that

$$\phi(t) = \text{length}(\gamma|_{[0,t]})$$

is a continuous function.

3. Let $\gamma : [0,1] \to \mathbb{C}$ be a rectifiable curve of length L. Suppose also that there is no (non-empty) open interval $(a,b) \subset [0,1]$ on which γ is constant. Show that the function ϕ defined by

$$\ell(t) = \text{length}(\gamma|_{[0,t]})$$

is a homeomorphism of $[0,1]$ onto $[0,L]$. This shows that such curves admit an arc-length reparametrization.

4. Let $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and let a, b, c, d be real numbers obeying $ad - bc = 1$.
(a) Show that $\phi : z \mapsto (az+b)/(cz+d)$ is a homeomorphism of $\mathbb{H}$ to itself.
(b) Show that 'hyperbolic area' is invariant under ϕ: For any $f \in C_c^\infty(\mathbb{H})$,

$$\iint_{\mathbb{H}} f \circ \phi(x+iy) \frac{dx\,dy}{y^2} = \iint_{\mathbb{H}} f(x+iy) \frac{dx\,dy}{y^2}$$

(c) For a smooth curve $\gamma : [0,1] \to \mathbb{H}$ we define the hyperbolic length of γ by

$$\text{len}_{\mathbb{H}}(\gamma) = \int_0^1 |\dot\gamma(t)| \frac{dt}{\operatorname{Im}\gamma(t)}$$

Show that $\text{len}_{\mathbb{H}}$ is also invariant under ϕ (i.e., $\text{len}_{\mathbb{H}}(\phi \circ \gamma) = \text{len}_{\mathbb{H}}(\gamma)$)
(d) Compute the hyperbolic length of a line segment from i to iy for general $y > 0$ and show that no other smooth path has less length.
(e) Show that the shortest (in the hyperbolic sense) smooth path between two points in $\mathbb{H}$ that are not in a vertical line is an arc of a circle with center on the x-axis.
[*Note:* In (d) and (e) you are **not** required to classify (or even discuss) the full class of smooth paths that achieve the minimal length. They are reparameterizations of the curves you find, but we have to expand the notion of reparameterization to allow for curves with intervals of constancy — all in all, more trouble than it is worth.]

5. For $j \in \{1,2\}$, let $\gamma_j : (-1,1) \to \mathbb{R}^n$ be C^1 curves. Suppose also that $\gamma_1(0) = \gamma_2(0) =: z_0$, and that $\gamma_j'(0) \neq 0$ for both curves. We define the angle between the two curves as that $\theta \in [0,\pi]$ so that

$$\frac{\langle \gamma_1'(0), \gamma_2'(0)\rangle_{\mathbb{R}^n}}{\|\gamma_1'(0)\|\,\|\gamma_2'(0)\|} = \cos\theta.$$

We extend this notion to $\mathbb{C}$ via the usual identification with $\mathbb{R}^2$.

(a) Show that the inverse stereographic projection

$$(x, y) \in \mathbb{R}^2 \mapsto \tfrac{1}{1+x^2+y^2}\big(2x, 2y, x^2 + y^2 - 1\big) \in \mathbb{R}^3,$$

is a conformal (=angle preserving) map.
(b) Find a smooth bijection $\phi : \mathbb{R} \to (0, \pi)$ with $\phi(0) = \frac{\pi}{2}$ so that

$$F : (x, y) \mapsto \begin{pmatrix} \cos(x)\sin(\phi(y)) \\ \sin(x)\sin(\phi(y)) \\ \cos(\phi(y)) \end{pmatrix}$$

is conformal. This is the inverse of the Mercator projection.