Realizable Extensions of Intuitionistic Analysis: Brouwer, Kleene, Kripke and the End of Time

Joan Rand Moschovakis

Occidental College, Emerita

ASL Special Session on Intuitionism and Intuitionistic Logic
APA Pacific Division Spring Meeting, Seattle
April 12, 2017

- Real numbers can be represented by convergent sequences of rationals (coded by infinite sequences of natural numbers).
- Brouwer considered a *reduced continuum* of completed *lawlike sequences* and a *full continuum*, of *potentially infinite arbitrary choice sequences*, for which the law of excluded middle fails.
- In a recent lecture Kripke suggested viewing the intuitionistic full continuum as an expansion *in time* of the classical continuum, depending on the activity of a creating subject.
- Question: Would there be *one* continuum *at the end of time*?
- Classical analysis **C** (with countable choice) is classically equivalent to its negative translation $\mathbf{C}^\circ$, and $\mathbf{C}^\circ$ is consistent with Kleene's 2-sorted system **I** of intuitionistic analysis.
- **I** and **C** prove "not every sequence is recursive," but **I** is *consistent with* "there are no non-recursive sequences."
- Theorem: A 3-sorted extension **IC** of **I** and $\mathbf{C}^\circ$, with an *end of time* axiom ET: "there are no non-classical sequences," is consistent. "Not every sequence is classical" is *independent of* **IC**.

The development of Brouwer's intuitionistic analysis

In his dissertation "On the foundations of mathematics" (1907), L. E. J. Brouwer agreed with Kant that the intuition of time is *a priori*, and asserted that the possibility of distinguishing moments in time was at the base of all mathematical reasoning.

Brouwer's construction of mathematics began with the positive integers 1, 2, 3, . . . , obtained by repeatedly taking the successor; then 0, -1, -2, . . . ("continu[ing] the sequence of ordinal numbers to the left"), then the rational numbers (pairs of ordinal numbers) and "the usual irrationals (first of all the expressions containing fractional exponents) . . . as symbolic aggregates of previously introduced numbers . . ." so "at each stage of development of the theory the set of the numbers known remains denumerable."

From the rationals on, the set of numbers known at each stage is *"everywhere dense in itself,"* but never constitutes a continuum.

The dual fractions (rationals $\pm m/2^n$) form a denumerable dense "scale . . . constructed on the continuum," of the order type of the rationals. Every point of the continuum can be approximated arbitrarily closely by a sequence of dual fractions but ". . . we can never consider the approximating sequence of a *given definite point* as being *completed*, so we must consider it as partly unknown." "From the fact that every conceivable approximating sequence occurs it can be deduced, following Cantor . . . , that it is impossible to enumerate all the points of the continuum."

In 1907 Brouwer relied on "the intuition of continuity, of 'fluidity'" to pass from the dual fractions to the "measurable continuum," constituting "a matrix of points to be considered as a whole."

He allowed classical reasoning about the measurable continuum. Every bounded infinite set of points "has at least one limit point . . . otherwise there would be a shortest distance between points."

In 1908 (the year after his dissertation) Brouwer published "The unreliability of the logical principles," arguing against the unrestricted use of the law of excluded middle (LEM).

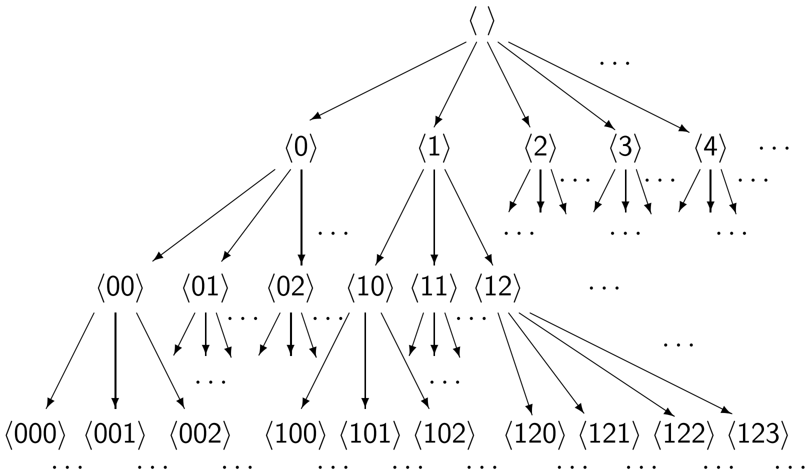
Ten years later he returned to the problem of constructing the continuum, this time avoiding the LEM and admitting infinitely proceeding sequences or "choice sequences" of natural numbers as mathematical objects of a new kind. Every sequence of dual fractions approximating a point on the measurable continuum is expressible as a choice sequence of numbers coding dual fractions.

An arbitrary choice sequence α is *potentially infinite* but may be unfinished. At any stage in the construction of α only a finite initial segment $\overline{\alpha}(n) = \langle \alpha(0), \ldots, \alpha(n \dot{-} 1) \rangle$ may have been determined.

Restrictions on further choices may be made at any stage. If *all* subsequent choices are specified, the sequence becomes "lawlike."

Every lawlike sequence is a choice sequence. But unless there are uncountably many "laws," not all choice sequences can be lawlike.

Classical *Baire space* is the set ω^ω of all infinite sequences of natural numbers with the finite initial segment topology. Brouwer's *universal spread* is intuitionistic Baire space. His choice sequences are potentially infinite branches of an ω-branching tree:



Brouwer's view of choice sequences as only *potentially* infinite led him to accept the following principles, as interpreted by Kleene:

- Countable choice: If for every natural number n there is a choice sequence α with the property $A(n, \alpha)$, then there is a choice sequence β with the property that for each n the property $A(n, \beta_n)$ holds, where $\beta_n(m) = \alpha(2^n \cdot 3^m)$ for all m.
- "The Bar Theorem": A "thin bar" $B(w)$ is a property of finite sequences (*nodes* of the tree) such that every choice sequence α has exactly one initial segment $\overline{\alpha}(n)$ satisfying $B(\overline{\alpha}(n))$. If $A(w)$ is a property of finite sequences such that
    - $B(w)$ implies $A(w)$, and
    - "*A propagates back across the nodes*": if $A(u)$ holds of every immediate successor u of w, then $A(w)$ holds also,

  then $A(w)$ holds for every node above the bar, including $\langle\,\rangle$.
- Continuous choice: If for every choice sequence α there is a choice sequence β with the property $A(\alpha, \beta)$, then there is a *continuous* function F from choice sequences to choice sequences such that $A(\alpha, F(\alpha))$ holds for all α.

The Bar Theorem is derivable from countable choice using classical logic, so both are classically reasonable assumptions.

Continuous choice, on the other hand, is not. Brouwer's intuitionistic analysis proves $\forall\alpha\neg\neg[\forall x\alpha(x) = 0 \vee \neg\forall x\alpha(x) = 0]$, but also $\neg\forall\alpha[\forall x\alpha(x) = 0 \vee \neg\forall x\alpha(x) = 0]$, contradicting the LEM.

If Brouwer could also prove $\forall\alpha[\forall x\alpha(x) = 0 \vee \neg\forall x\alpha(x) = 0]$ he would contradict himself. But intuitionistic logic replaces the classical principle $\neg\neg A \rightarrow A$ by $\neg A \rightarrow (A \rightarrow B)$ (*ex falso sequitur quodlibet*). Kleene developed a formal system **I** of intuitionistic analysis and established its consistency by function realizability.

Intuitionistic negation expresses *inconsistency*: $\neg A \equiv A \rightarrow 0 = 1$.

Intuitionistic double negation expresses *persistent consistency*: If $\neg\neg A$ holds, then $\neg A$ is inconsistent with the present state of knowledge, so A will be consistent with every consistent extension of the present state of knowledge. But $\neg\neg A$ does not entail A.

Intuitionistic analysis **I** vs. classical analysis **C**:

Classical analysis **C** has classical logic and all the mathematical axioms of Kleene's intuitionistic analysis **I** *except* the Bar Theorem (derivable classically from countable choice) and continuous choice (classically false). First, some reassuring similarities:

- **I** and **C** prove the same Π^0_2 sentences, so they have the same provably recursive functions.
- If either **I** or **C** is consistent, so is the other. The proof uses function-realizability with Gödel's negative interpretation.
- Classical first-order arithmetic is consistent with **I** by classical relativized function-realizability.
- Both **I** and **C** prove that every function which is continuous on a compact interval is uniformly continuous there.
- The Heine-Borel Theorem and the Jordan Curve Theorem can be formulated and proved in **I**.

The usual statements of the Bolzano-Weierstrass and Intermediate Value Theorems fail intuitionistically. Constructive versions are awkward to use. But a classical proof of an existential statement usually does not provide a witness (Σ^0_1 statements are the exception) while an intuitionistic proof of *any* existential statement provides a *recursive* witness, by recursive function realizability.

Classical reasoning can be expressed intuitionistically without $\vee$ and $\exists$, by the Gödel-Gentzen negative interpretation. We can let $\mathrm{A} \overset{\circ}{\vee} \mathrm{B} \equiv \neg(\neg \mathrm{A} \,\&\, \neg \mathrm{B})$ and $\exists^\circ \ldots \equiv \neg\forall \ldots \neg$ for easier reading. Constructive $\vee$ and $\exists$ retain their stronger interpretations.
The negative translation of the countable axiom of choice:

$$\mathsf{AC}^\circ_{01}: \qquad \forall \mathrm{x} \exists^\circ \alpha \mathrm{A}^\circ(\mathrm{x}, \alpha) \rightarrow \exists^\circ \beta \forall \mathrm{x} \mathrm{A}^\circ(\mathrm{x}, \lambda \mathrm{y}.\beta(2^{\mathrm{x}} \cdot 3^{\mathrm{y}}))$$

is consistent with $\mathbf{I}$ by realizability, so intuitionistic and (negatively expressed) classical reasoning can peacefully coexist in $\mathbf{I} + \mathsf{AC}^\circ_{01}$. But in order to compare choice sequences with classical lawlike sequences *as objects*, *two* sorts of sequence variables are required.

C$^\circ$ is a *negative* version of classical analysis with countable choice, in a language $\mathcal{L}(\mathbf{C}^\circ)$ with variables $\mathrm{i}, \mathrm{j}, \ldots, \mathrm{q}, \mathrm{w}, \mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{i}_1, \ldots$ over natural numbers and $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{a}_1, \ldots$ over sequences of numbers; constants for $0, ', +, \cdot$ and additional primitive recursive functions as needed; Church's λ; equality $=$ for numbers; parentheses, also denoting function application; and the logical constants $\&, \neg, \rightarrow, \forall$.

C$^\circ$-*terms* (type 0) and C$^\circ$-*functors* (type 1) are defined inductively. Number variables and 0 are C$^\circ$-*terms*. Sequence variables and $'$ are C$^\circ$-*functors*. If u is a C$^\circ$-*functor* and t is a C$^\circ$-*term* then $(\mathrm{u})(\mathrm{t})$ (also written $\mathrm{u}(\mathrm{t})$) is a C$^\circ$-*term* and $\lambda\mathrm{x}(\mathrm{t})$ (also written $\lambda\mathrm{x}.\mathrm{t}$) is a C$^\circ$-*functor*. If s, t are C$^\circ$-*terms* and u is a C$^\circ$-*functor* then $\mathrm{s} + \mathrm{t}$, $\mathrm{s} \cdot \mathrm{t}$, s^{t}, p_{s} (the s^{th} prime), $\Sigma_{\mathrm{i} \leq \mathrm{t}}\mathrm{u}(\mathrm{i})$, $\Pi_{\imath \leq \mathrm{t}}\mathrm{u}(\mathrm{i}), \ldots$ are C$^\circ$-*terms*.

If s, t are C$^\circ$-*terms* then $(\mathrm{s} = \mathrm{t})$ is a *prime formula*. If A, B are *formulas* so are $(\mathrm{A} \,\&\, \mathrm{B}),\ (\mathrm{A} \rightarrow \mathrm{B}),\ (\neg\mathrm{A}),\ (\forall\mathrm{x}\mathrm{A}), (\forall\mathrm{b}\mathrm{A})$.

The logical rules and axioms for $\&, \neg, \rightarrow, \forall\mathrm{x}, \forall\mathrm{b}$ are intuitionistic, e.g. $\forall\mathrm{b}\mathrm{A}(\mathrm{b}) \rightarrow \mathrm{A}(\mathrm{u})$ where u is a C$^\circ$-functor free for b in $\mathrm{A}(\mathrm{b})$.

Mathematical axioms of $\mathbf{C}^\circ$:

- $=$ is an equivalence relation, $\mathrm{x} = \mathrm{y} \to \mathrm{a}(\mathrm{x}) = \mathrm{a}(\mathrm{y})$, 0 is not a successor, and $'$ is one-to-one.
- Primitive recursive defining equations for function constants.
- Mathematical induction: $\mathrm{A}(0) \,\&\, \forall \mathrm{x}(\mathrm{A}(\mathrm{x}) \to \mathrm{A}(\mathrm{x}')) \to \mathrm{A}(\mathrm{x})$ for formulas $\mathrm{A}(\mathrm{x})$ of $\mathcal{L}(\mathbf{C}^\circ)$.
- λ-reduction: $(\lambda \mathrm{x}.\mathrm{r}(\mathrm{x}))(\mathrm{t}) = \mathrm{r}(\mathrm{t})$ for C°-terms $\mathrm{r}(\mathrm{x}), \mathrm{t}$.
- Negative axiom of countable choice for formulas A of $\mathcal{L}(\mathbf{C}^\circ)$:

$$\mathsf{AC}_{01}^{\mathbf{C}^\circ} : \forall \mathrm{x} \exists^\circ \mathrm{a} \mathrm{A}(\mathrm{x}, \mathrm{a}) \to \exists^\circ \mathrm{b} \forall \mathrm{x} \mathrm{A}(\mathrm{x}, \lambda \mathrm{y}.\mathrm{b}(2^{\mathrm{x}} \cdot 3^{\mathrm{y}})).$$

Proposition. $\mathbf{C}^\circ \vdash \neg\neg \mathrm{A} \to \mathrm{A}$ for formulas A of $\mathcal{L}(\mathbf{C}^\circ)$.

Proposition. Let $\mathbf{B}$ be the classically correct subsystem of $\mathbf{I}$ which only omits continuous choice, and let $\mathbf{C} = \mathbf{B} + \neg\neg \mathrm{A} \to \mathrm{A}$. There is a faithful negative translation $\mathrm{A} \mapsto \mathrm{A}^{\mathrm{tr}}$ from $\mathbf{C}$ to $\mathbf{C}^\circ$.

Finally, **IC** is a *three-sorted* formal system combining **I** and $\mathbf{C}^\circ$, adding existential quantifiers $\exists \mathrm{b}$ over classical sequences, with intuitionistic logic throughout, and with an *end of time* axiom

$$\text{ET:} \quad \forall\alpha\neg\neg\exists \mathrm{b}\forall \mathrm{x}\, \alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$$

(or $\forall\alpha\neg\forall \mathrm{b}\neg\forall \mathrm{x}\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$, abbreviated $\forall\alpha\exists^\circ \mathrm{b}\forall \mathrm{x}\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$).

I and $\mathbf{C}^\circ$ have the same primitive recursive function constants. Both classical sequence variables $\mathrm{a}, \mathrm{b}, \ldots$ and choice sequence variables $\alpha, \beta, \ldots$ are now *functors*. Terms or functors without choice sequence variables are C-*terms* or C-*functors* respectively.

The logical axioms and rules of **I** are extended to the three-sorted language $\mathcal{L}(\mathbf{IC})$. Intuitionistic axioms and rules for $\forall \mathrm{b}$ and $\exists \mathrm{b}$ are added, e.g.: $\mathrm{A}(\mathrm{u}) \to \mathrm{B}$ / $\exists \mathrm{b}\mathrm{A}(\mathrm{b}) \to \mathrm{B}$, where u is a C-functor free for b in $\mathrm{A}(\mathrm{b})$ and b is not free in B. The mathematical axioms of **IC** are those of **I** extended to $\mathcal{L}(\mathbf{IC})$, plus $\mathsf{AC}_{01}^{\mathbf{C}^\circ}$ (only for negative C-formulas, as in $\mathbf{C}^\circ$), plus ET.

Proposition. $\mathbf{IC} \vdash \forall \mathrm{b}\exists\alpha\forall \mathrm{x}\mathrm{b}(\mathrm{x}) = \alpha(\mathrm{x})$.

From now on, we assume $\mathcal{M} = (\omega, \mathcal{C})$ is a classical ω-model of **C**.

Using $\mathcal{M}$ we can define a (modified) ${}^{\mathcal{C}}$*realizability interpretation*, using elements of $\mathcal{C}$ as the *actual* ${}^{\mathcal{C}}$realizing objects and to interpret free sequence variables of both sorts. The *potential* ${}^{\mathcal{C}}$realizing objects, and interpretations of free *choice sequence* variables in the corresponding definition, are elements of ω^ω.
A *sentence* of $\mathcal{L}(\mathbf{IC})$ is ${}^{\mathcal{C}}$*realizable* if and only if it has a *recursive* ${}^{\mathcal{C}}$realizer, and a formula is ${}^{\mathcal{C}}$*realizable* if its universal closure is.
$\mathcal{C}$ is recursively closed since $\mathcal{M}$ is an ω-model of **C**.

Lemma. For every negative C-formula E of $\mathcal{L}(\mathbf{IC})$ with only Ψ free there is a primitive recursive potential ${}^{\mathcal{C}}$realizer τ_{E} for E such that for each interpretation $\mathit{\Psi}$ of Ψ by elements of $\mathcal{C}$ and ω:

1. If E is ${}^{\mathcal{C}}$realized-$\mathit{\Psi}$ by some $\varepsilon \in \mathcal{C}$ then E is true-$\mathit{\Psi}$ in $\mathcal{M}$.
2. If E is true-$\mathit{\Psi}$ in $\mathcal{M}$ then τ_{E} ${}^{\mathcal{C}}$realizes-$\mathit{\Psi}$ E.

A sentence E of $\mathcal{L}(\mathbf{C}^\circ)$ is ${}^{\mathcal{C}}$realizable if and only if E is true in $\mathcal{M}$.

Theorem. If $\mathrm{F}_1, \ldots, \mathrm{F}_\mathrm{n}, \mathrm{E}$ are formulas of $\mathcal{L}(\textbf{IC})$ such that $\mathrm{F}_1, \ldots, \mathrm{F}_\mathrm{n} \vdash_{\textbf{IC}} \mathrm{E}$ and $\mathrm{F}_1, \ldots, \mathrm{F}_\mathrm{n}$ are all ${}^{\mathcal{C}}$realizable, then E is ${}^{\mathcal{C}}$realizable. Since $0 = 1$ is not ${}^{\mathcal{C}}$realizable, **IC** is consistent.

Corollary. **IC** $+$ NegTh($\mathcal{M}$) is consistent, where NegTh($\mathcal{M}$) is the set of all sentences of $\mathcal{L}(\textbf{C}^\circ)$ which are true in $\mathcal{M}$.

Theorem. $\forall\alpha\exists \mathrm{b}\forall \mathrm{x}\, \alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$ is independent of **IC**.

Proof. If $\mathcal{C} = \omega^\omega$ then $\forall\alpha\exists \mathrm{b}\forall \mathrm{x}\, \alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$ is ${}^{\mathcal{C}}$realizable, so **IC** $\nvdash \neg\forall\alpha\exists \mathrm{b}\forall \mathrm{x}\, \alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$, so **IC** $+ \forall\alpha\exists \mathrm{b}\forall \mathrm{x}\, \alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$ is consistent. If $\mathcal{C} \neq \omega^\omega$ then $\neg\forall\alpha\exists \mathrm{b}\forall \mathrm{x}\, \alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$ is ${}^{\mathcal{C}}$realizable, so **IC** $\nvdash \neg\neg\forall\alpha\exists \mathrm{b}\forall \mathrm{x}\, \alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$, so **IC** $+ \neg\forall\alpha\exists \mathrm{b}\forall \mathrm{x}\, \alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$ is also consistent (assuming **C** has a *proper* ω-model).

Corollary. If $\mathcal{M}$ is a proper ω-model of **C**, then **IC** $+ \neg\forall\alpha\exists \mathrm{b}\forall \mathrm{x}\, \alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x}) +$ NegTh($\mathcal{M}$) is consistent.

Remark. By relativizing to ${}^{\mathcal{C}}$realizability$/\mathcal{C}$, all these results remain true when all classically true sentences of arithmetic are added.

Markov's Principle MP_1: $\neg\neg\exists \mathrm{x}\alpha(\mathrm{x}) = 0 \rightarrow \exists \mathrm{x}\alpha(\mathrm{x}) = 0$ is not ${}^{\mathcal{C}}$realizable/$\mathcal{C}$, so $\neg\forall\alpha[\neg\neg\exists \mathrm{x}\alpha(\mathrm{x}) = 0 \rightarrow \exists \mathrm{x}\alpha(\mathrm{x}) = 0]$ is consistent with **IC**. By Vesley's work, **IC** is consistent with Brouwer's creating subject counterexamples because the *independence of premise* principle IP: $(\neg \mathrm{A} \rightarrow \exists\beta \mathrm{B}(\beta)) \rightarrow \exists\beta(\neg \mathrm{A} \rightarrow \mathrm{B}(\beta))$ is ${}^{\mathcal{C}}$realizable.

A weaker version $\neg\neg\exists\beta[\forall \mathrm{x}\beta(\mathrm{x}) = 0 \leftrightarrow \neg \mathrm{A}]$ (with β not free in A) of (weak) Kripke's schema is ${}^{\mathcal{C}}$realizable, so consistent with **IC**.

Interpretation. Even if the creating subject (working according to just the principles of **IC**) can prove all classically true arithmetical sentences (including Markov's Principle for *recursive* sequences) and all true negative sentences about classical sequences,

- by ET, the creating subject will not be able to construct a choice sequence which differs from every classical sequence;
- the creating subject will be unable to decide if every choice sequence is extensionally equal to a classical sequence or not.

Some References:

1. Brouwer, L. E. J., "Essentially negative properties," 1948A in *L. E. J. Brouwer Collected Works 1*, ed. Heyting, p. 478.
2. Kleene, S. C. and Vesley, R. E., *Foundations of Intuitionistic Mathematics, Especially in Relation to Recursive Functions*, North-Holland, Amsterdam 1965.
3. Kripke, S., slides from Brouwer Symposium, Amsterdam 2016.
4. Moschovakis, J. R., "Can there be no non-recursive functions?," Journal of Symbolic Logic, 1971.
5. Moschovakis, J. R., "Intuitionistic analysis at the end of time," submitted 2017.
6. Niekus, J., "What is a choice sequence?," preprint.
7. Troelstra, A. S., *Choice Sequences, a Chapter of Intuitionistic Mathematics*, Clarendon Press, Oxford 1977.