# Intuitionistic Analysis, Forward and Backward

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In the early 20th century the Dutch mathematician L. E. J. Brouwer questioned the universal applicability of the Aristotelian law of excluded middle and proposed basing mathematical analysis on informal intuitionistic logic, with natural numbers and *choice sequences* (infinitely proceeding sequences of more or less freely chosen natural numbers) as individual objects.

For Brouwer, numbers and choice sequences were mental constructions which by their nature satisfied

- mathematical induction,
- countable and dependent choice,
- bar induction,
- and a continuity principle contradicting classical logic.

Brouwer disparaged efforts to justify mathematical theories by proving consistency, but there is a consistency question here. In 1930 A. Heyting formalized intuitionistic logic and arithmetic, and attempted a complex formalization of intuitionistic analysis.

In 1965 S. C. Kleene and R. E. Vesley developed Brouwer's intuitionistic analysis in a formal system **FIM** extending intuitionistic arithmetic, with quantifiers and variables over natural numbers and one-place number-theoretic functions, constants and axioms for finitely many primitive recursive functions and functionals, and two-sorted intuitionistic logic.

Kleene's function-realizability interpretation proved **FIM** consistent relative to its classically correct basic subsystem **B**, and clarified the relationship between intuitionistic and classical analysis:

- The mathematical axioms of **B** include full mathematical induction, countable choice, and restricted bar induction.
- Classical analysis **C** is like **B** but with classical logic (which incidentally removes the restriction on bar induction).
- Intuitionistic analysis **FIM** is **B** together with a strong axiom of continuous choice ("Brouwer's Principle for Functions").

*Logic and Language*:

- Connectives $\&, \vee, \neg, \rightarrow$, and quantifiers $\forall, \exists$ of both sorts.
- The classical axioms $\neg\neg A \rightarrow A$ and $A \vee \neg A$ are replaced by $\neg A \rightarrow (A \rightarrow B)$ and $(A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)$.
- *Type-0 (number) variables* are $a, b, \ldots, x, y, z, a_1, \ldots$
- *Prime formulas* are equations $s = t$ between *terms* of type 0.

The second sort of variables could range over sets ("species") or sequences. *Detachable* species have characteristic functions, so

- *Type-1 variables* $\alpha, \beta, \gamma, \ldots, \alpha_1, \ldots$ range over sequences.
- *Schemas* express properties of *definable* species $A(\alpha, x)$.
- Equality is *extensional*: $\alpha = \beta$ abbreviates $\forall x(\alpha(x) = \beta(x))$.

*Peculiar characteristics of intuitionistic logic*:

- $(A \rightarrow B) \rightarrow \neg(A \,\&\, \neg B)$ is provable, but the converse is not.
- $\exists x \neg A(x) \rightarrow \neg\forall x A(x)$ is provable, but the converse is not.
- $\neg\neg(A \vee \neg A)$ is provable, but $\neg\neg\forall x(A(x) \vee \neg A(x))$ is not.

$\mathbf{IA}_1$ is *2-sorted intuitionistic arithmetic* with $\lambda$-abstraction and finitely many constants for primitive recursive functions and functionals. The mathematical axioms of $\mathbf{IA}_1$ are

- the usual axioms for $=$, 0, $'$, $+$, $\cdot$
- defining equations for the other function constants
- mathematical induction for all formulas $A(x)$:
$$A(0) \mathbin{\&} \forall x(A(x) \to A(x')) \to \forall x A(x).$$
- $(x = y \to \alpha(x) = \alpha(y))$ and
- $\lambda$-reduction: $(\lambda x.u(x))\,(s) = u(s)$ where $u(x)$, $s$ are terms.

*Easy Facts*:

- $\mathbf{IA}_1$ proves $\forall x \forall y(x = y \vee \neg(x = y))$.
- In fact, if $A(x)$ is quantifier-free or has only bounded number quantifiers then $\mathbf{IA}_1$ proves $\forall x(A(x) \vee \neg A(x))$.
- There is a classical model of $\mathbf{IA}_1$ in which the sequence variables range over all primitive recursive functions.

Brouwer accepted *countable choice*:

$$\mathbf{AC}_{01}: \quad \forall x \exists \alpha A(x, \alpha) \to \exists \beta \forall x A(x, \lambda y \beta(2^x \cdot 3^y))$$

and *bar induction* with a detachable bar:

$$\mathbf{BI}_1: \quad \forall \alpha \exists x \rho(\overline{\alpha}(x)) = 0 \ \& \ \forall w[\rho(w) = 0 \to A(w)]$$
$$\& \ \forall w[\forall s A(w * \langle s \rangle) \to A(w)] \to A(\langle\,\rangle)$$

where $w$ ranges over codes for finite sequences of numbers, $\langle\,\rangle = \overline{\alpha}(0)$ codes the empty sequence, $\langle \alpha(0), \ldots, \alpha(x) \rangle = \overline{\alpha}(x+1)$ codes the first $x+1$ values of $\alpha$, and $*$ expresses concatenation.

Bar induction is backwards induction on the nodes of a tree with every branch finite but with countably infinite branching, from the tips of the branches (where $\rho(\overline{\alpha}(x)) = 0$ so $A$ holds), back along the nodes (if $A$ holds at every immediate successor of a node $w$, then $A$ holds at $w$), proving finally that $A$ holds at the root $\langle\,\rangle$.

★ Kleene's neutral *basic analysis* is $\mathbf{B} = \mathbf{IA}_1 + \mathrm{AC}_{01} + \mathrm{BI}_1$.

*Observations*:

- Although formulated with intuitionistic logic, Kleene's basic analysis **B** is logically *neutral* in the sense that all its theorems are also correct with classical logic.
- From $AC_{01}$ follows $\mathbf{AC}_{00}$: $\forall x \exists y A(x, y) \to \exists \alpha \forall x A(x, \alpha(x))$.
- Bar induction justifies transfinite recursion up to a countable ordinal, using the *Kleene-Brouwer ordering* on finite sequences.
- With classical logic, bar induction justifies *König's Lemma*: Every finitely branching tree with arbitrarily long finite branches has an infinite branch.
- With intuitionistic logic, bar induction justifies the *Fan Theorem*: If a finitely branching tree has only finite branches, there is a finite upper bound $n$ to the lengths of the branches.

The Fan Theorem is a strong generalization of the fact that there is a longest possible game of chess under the usual rules.

Brouwer's distinctive contribution to intuitionistic analysis, his *continuity principle*, followed from his acceptance of *arbitrary choice sequences* as legitimate mathematical objects. He distinguished between "fundamental sequences, i.e. predeterminate infinite sequences which like classical ones, proceed in such a way that, from the beginning, the *m*th term is fixed for each *m*" and "infinitely proceeding sequences $p_1, p_2, \ldots$ whose terms are chosen more or less freely from mathematical entities previously acquired; in such a way that the freedom of choice existing perhaps for the first element $p_1$ may be subjected to a lasting restriction at some following $p_i$ and again and again to sharper lasting restrictions ..."

He reasoned that any function defined on *all* choice sequences $\alpha$ of natural numbers must be *continuous*, because it assigns values to sequences for which only a finite initial segment may have been determined so far. In particular, if $\Phi\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ then

$$\forall\alpha\exists x\forall\beta(\overline{\alpha}(x) = \overline{\beta}(x) \to \Phi(\alpha) = \Phi(\beta)).$$

In the two-sorted language of analysis this continuity principle can be expressed only for *definable* functions $\Phi$: $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$.

Let $\exists! x B(x)$ abbreviate $\exists x B(x)$ & $\forall x \forall y(B(x)$ & $B(y) \to x = y)$

so $\forall \alpha \exists! y A(\alpha, y)$ expresses that $A$ defines a function from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$.

Brouwer's *weak continuity principle* can then be stated

**WC**$_{10}$: $\quad \forall \alpha \exists! y A(\alpha, y) \to \forall \alpha \exists x \exists y \forall \beta(\overline{\alpha}(x) = \overline{\beta}(x) \to A(\beta, y))$.

If the $y$ is not required to be unique and the output includes a modulus of continuity, a *principle of continuous choice* results:

$$\mathbf{CC}_{10}: \quad \forall \alpha \exists y A(\alpha, y) \to$$
$$\exists \sigma \forall \alpha [\exists y \{\sigma\}(\alpha) \simeq y \,\&\, \forall y(\{\sigma\}(\alpha) \simeq y \to A(\alpha, y))],$$

where $\{\sigma\}(\alpha) \simeq y \equiv \exists z[\sigma(\overline{\alpha}(z)) = y + 1 \ \&\ \forall k < z \ \sigma(\overline{\alpha}(k)) = 0]$.

Continuous choice for functions from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}^{\mathbb{N}}$ is stronger:

$$\mathbf{CC}_{11}: \quad \forall \alpha \exists \beta A(\alpha, \beta) \to$$
$$\exists \sigma \forall \alpha [\exists \beta \{\sigma\}[\alpha] \simeq \beta \ \&\ \forall \beta(\{\sigma\}[\alpha] \simeq \beta \to A(\alpha, \beta))].$$

★ Kleene's *Intuitionistic Analysis* is $\mathbf{FIM} = \mathbf{B} + \mathrm{CC}_{11}$.

Kleene removed the mystery from intutionistic analysis by proving

- **FIM** proves $\neg\forall\alpha(\forall x\alpha(x) = 0 \vee \neg\forall x\alpha(x) = 0)$.
  Even the weak continuity principle contradicts classical logic.
- **FIM** is consistent relative to its neutral subsystem **B**.
  The proof used *function-realizability*, a precise implementation of the "B-H-K interpretation" of intuitionistic logic.
- **FIM** can only prove the existence of *recursive* sequences.
  If $\forall x\exists y A(x, y)$ is a *closed theorem* of **FIM** then there is a gödel number $e$ for which **B** proves $\exists\alpha\forall x(\{\mathbf{e}\}(x) \simeq \alpha(x))$ and **FIM** proves $\forall\alpha[\forall x(\{\mathbf{e}\}(x) \simeq \alpha(x)) \to A(x, \alpha)]$.
- The recursive sequences do not form a classical model for **B**, as they do not satisfy the Fan Theorem.
- Even the *hyperarithmetical* sequences do not form a classical model for **B**, as they do not satisfy the Bar Theorem.

R. E. Vesley used **FIM** to analyze and formalize Brouwer's results on the structure of the intuitionistic real numbers.

Now in the 21st century, Wim Veldman and others are developing an *intuitionistic reverse analysis* parallel to, but diverging significantly from, both the classical reverse mathematics of H. Friedman and S. Simpson and the constructive reverse analysis being developed by followers of E. Bishop. To understand the differences, first consider the constructive and classical versions.

D. Bridges stated the goals of *constructive reverse mathematics* as

- to determine which constructive principles are needed, in the context of informal intuitionistic set theory, to prove particular theorems of Bishop's constructive mathematics, and
- to determine which additional nonconstructive principles would be needed to prove particular classical theorems.

If a result is obtained informally over **IZF**, some work may be needed to determine which axioms are strictly necessary for its proof. This seems undesirable for reverse mathematics; H. Ishihara recently proposed a precise, formal constructive reverse analysis.

The goal of *classical reverse mathematics* is to determine which set existence axioms are needed to prove a particular theorem of "ordinary" classical mathematics. S. Simpson distinguished five main subsystems of classical second-order arithmetic $\mathbf{Z}_2$ extending a fragment of Peano arithmetic with restricted induction, in a language with variables and quantifiers over numbers and sets of numbers, and with constants 0, 1, $+$, $\cdot$, $=$, $<$ and $\in$.

These subsystems have increasingly strong set existence axioms:

- $\mathbf{RCA}_0$ (recursive comprehension and $\Sigma^0_1$-induction),
- $\mathbf{WKL}_0$ ("weak König's Lemma"),
- $\mathbf{ACA}_0$ (arithmetical comprehension),
- $\mathbf{ATR}_0$ (arithmetical transfinite recursion),
- $\Pi^1_1$-$\mathbf{CA}_0$ ($\Pi^1_1$ comprehension).

The subscript 0 denotes *restricted mathematical induction*.

**RCA**, **WKL**, **ACA**, **ATR**, $\Pi^1_1$-**CA** are the corresponding systems with *unrestricted mathematical induction*.

Classical reverse mathematics uses the language of set theory but intuitionistic reverse analysis should use the language of analysis. Both implicitly assume a two-sorted primitive recursive arithmetic.

*Primitive recursive arithmetic* $\mathbf{PRA}_0$ is a quantifier-free subsystem of first-order arithmetic, with constants and axioms for $=$, 0, $'$ and all primitive recursive functions, and mathematical induction for quantifier-free formulas only. Since $x = y \vee \neg(x = y)$ is provable intuitionistically using quantifier-free induction, there is no difference between classical and intuitionistic $\mathbf{PRA}_0$.

In the context of $\mathbf{PRA}_0$, formulas with bounded quantifiers such as $\forall x < y \exists z < y(x + z = y)$ are considered to be quantifier-free.

A two-sorted version adds variables over number-theoretic functions, with axioms allowing the definition of functions by composition and primitive recursion. *Two-sorted intuitionistic primitive recursive arithmetic* $\mathbf{IA}_1$ adds full mathematical induction.

One way to classify subsystems of intuitionistic analysis would be by the strength of their countable choice axioms. When formalizing intuitionistic analysis, Kleene assumed full countable choice:

$\mathbf{AC}_{01}$: $\quad \forall x \exists \alpha A(x, \alpha) \rightarrow \exists \beta \forall x A(x, \lambda y . \beta(2^x \cdot 3^y))$,

although he rarely needed more than its consequence

$\mathbf{AC}_{00}$: $\quad \forall x \exists y A(x, y) \rightarrow \exists \alpha \forall x A(x, \alpha(x))$.

$\mathrm{AC}_{00}$ is equivalent to the conjunction of a *bounded choice schema*:

$\mathbf{BC}_{00}$: $\quad \forall \mathrm{x} \exists \mathrm{y} \leq \beta(\mathrm{x}) \mathrm{A}(\mathrm{x}, \mathrm{y}) \rightarrow \exists \alpha \forall \mathrm{x} \mathrm{A}(\mathrm{x}, \alpha(\mathrm{x}))$

and a *bounding axiom schema*:

$\mathbf{AB}_{00}$: $\quad \forall x \exists y A(x, y) \rightarrow \exists \beta \forall x \exists y \leq \beta(x) A(x, y)$.

*Quantifier-free countable choice* is

**qf-**$\mathbf{AC}_{00}$: $\quad \forall x \exists y \alpha(x, y) = 0 \rightarrow \exists \beta \forall x \alpha(x, \beta(x)) = 0$.

★ *Intuitionistic recursive analysis* is $\mathbf{IRA} = \mathbf{IA}_1 + \text{qf-AC}_{00}$.

Kleene formalized recursive functional theory using the system $\mathbf{M}_1 = \mathbf{IA}_1 + \mathrm{AC}_{00}!$ with a comprehension ("unique choice") axiom

$$\mathbf{AC}_{00}!: \quad \forall x \exists ! y A(x, y) \rightarrow \exists \alpha \forall x A(x, \alpha(x))$$

where $\exists!$ says "there is exactly one." G. Vafeiadou observed that $\mathrm{AC}_{00}!$ is equivalent over $\mathbf{IA}_1$ to the conjunction of qf-$\mathrm{AC}_{00}$ and

$$\mathbf{CF}_0: \quad \forall x (A(x) \vee \neg A(x)) \rightarrow \exists \chi \forall x (\chi(x) = 0 \leftrightarrow A(x)),$$

which says that every detachable subset of the natural numbers has a characteristic function. With classical logic $\mathrm{CF}_0$ gives full comprehension, but with intuitionistic logic it is much weaker.

In the following diagram all arrows are strict with one exception: The question if $\mathbf{M}_1 + \mathrm{BC}_{00} \vdash \mathrm{AC}_{00}$ seems to be open. J. van Oosten proved $\mathbf{M}_1 + \mathrm{AB}_{00} \nvdash \mathrm{AC}_{00}$. The other independence results are due to S. Weinstein, G. Vafeiadou and JRM.

$$
\begin{tikzcd}
 & & \mathbf{M}_1 + \mathsf{AC}_{00} \arrow[dl] \arrow[dr] & & \\
 & \mathbf{M}_1 + \mathsf{BC}_{00} \arrow[dl] \arrow[dr] & & \mathbf{M}_1 + \mathsf{AB}_{00} \arrow[dl] \arrow[dr] & \\
\mathbf{IA}_1 + \mathsf{BC}_{00} \arrow[dr] & & \mathbf{M}_1 = \mathbf{IA}_1 + \mathsf{AC}_{00}! \arrow[dl] \arrow[dr] & & \mathbf{IA}_1 + \mathsf{AB}_{00} \arrow[dl] \\
 & \mathbf{IA}_1 + \mathsf{CF}_0 \arrow[dr] & & \mathbf{IA}_1 + \text{qf-}\mathsf{AC}_{00} \arrow[dl] & \\
 & & \mathbf{IA}_1 & &
\end{tikzcd}
$$

Most of these distinctions depend essentially on intuitionistic logic.

Let $\mathbf{IA}_1^c = \mathbf{IA}_1 + (A \vee \neg A)$ and $\mathbf{M}_1^c = \mathbf{M}_1 + (A \vee \neg A)$. Then

- $\mathbf{IA}_1^c + \mathrm{CF}_0$, for which the primitive recursively bounded functions form a classical $\omega$-model, is stronger than $\mathbf{IA}_1^c$ and weaker than $\mathbf{M}_1^c$.
- $\mathbf{IA}_1^c + \text{qf-AC}_{00}$, for which the recursive functions form a classical $\omega$-model, is stronger than $\mathbf{IA}_1^c$ and weaker than $\mathbf{M}_1^c$.
- $\mathbf{M}_1^c$ is equivalent to $\mathbf{M}_1^c + \mathrm{AC}_{00}$ so the rest of the diagram collapses.

There are only two obvious points of contact with classical reverse mathematics:

- Classical **RCA** corresponds to $\mathbf{IRA} = \mathbf{IA}_1 + \text{qf-AC}_{00}$.
- Classical **ACA** corresponds to the restriction $\mathrm{AC}_{00}^{-}!$ of intuitionistic "unique choice" $\mathrm{AC}_{00}!$ to arithmetical predicates.

Simpson's classical systems *with full induction* and their characteristic axioms, in order of increasing strength, are

- **RCA** (recursive comprehension axiom),
- **WKL** ("weak König's Lemma"),
- **ACA** (arithmetical comprehension axiom),
- **ATR** (arithmetical transfinite recursion),
- $\Pi^1_1$-**CA** ($\Pi^1_1$ comprehension axiom).

**WKL** should correspond to some version of the Fan Theorem.
I. Loeb, J. Berger and W. Veldman proved that the *binary fan theorem for a detachable predicate* is equivalent over **IRA** to

- the Heine-Borel Theorem for an enumerable open cover of $[0, 1]$ by intervals with rational endpoints,
- the statement that a pointwise continuous function on $[0, 1]$ with a modulus of continuity is uniformly continuous,
- and Brouwer's approximate fixed-point theorem for enumerable continuous functions on the unit square.

So it is reasonable to conclude that

- Classical **WKL** corresponds to **WFT** = **IRA** + $\Delta_0$-FT.

**ATR** should correspond to a version of bar induction. T. Coquand proved the Open Induction Principle for $2^{\mathbb{N}}$ and $[0,1]$ by *bar induction on a subtree of* $\mathbb{N}^{\mathbb{N}}$ *with at most binary branching*.

Veldman defined an *approximate fan* to be a tree in which each node has *at most finitely many* immediate successors, and proposed a new axiom, the *Approximate Fan Theorem*

**AFT**: If every branch of an explicit approximate fan is finite, the set of "leaves" (final nodes of the branches) is almost-finite.

He proved that AFT is equivalent over **IRA** to

- *Monotone enumerable bar induction* $\Sigma^0_1$-BI$^m$.
- Open Induction on $2^{\mathbb{N}}$ and $[0,1]$.
- The set of all sequences not yet secured by a detachable bar on $\mathbb{N}^{\mathbb{N}}$ is well-founded under the Kleene-Brouwer ordering.
- A contrapositive version of the Bolzano-Weierstrass theorem.

So it is perhaps reasonable to conclude that

- Classical **ATR** corresponds to **ABI** = **IAC** + $\Sigma^0_1$-BI$^m$.

We have proposed intuitionistic analogues of four of Simpson's five distinguished subsystems:

- **IRA** = **IA**$_1$ + qf-AC$_{00}$ as an intuitionistic analogue of **RCA**.
- **WFT** = **IRA** + $\Delta_0$-FT as an intuitionistic analogue of **WKL**.
- **IAC** = **WFT** + AC$^-_{00}$! as an intuitionistic analogue of **ACA**.
- **ABI** = **IAC** + $\Sigma^0_1$-BI$^m$ as an intuitionistic analogue of **ATR**.

Is there an intuitionistic analogue of $\Pi^1_1$-**CA**? A monotone choice principle $\mathbf{AC}^m_{1/2,0}$ is provable in **FIM**; over **B** it justifies monotone bar induction. Perhaps

- **MBI** = **ABI** + $\Pi^1_1$-AC$^m_{1/2,0}$ + $\Pi^1_1$-BI$_1$ corresponds to $\Pi^1_1$-**CA**?
- **IRA** + AC$_{01}$ + AC$^m_{1/2,0}$ + BI$_1$ corresponds to $\mathbf{Z}_2$?

This is work in progress and many questions remain.