INTUITIONISM AND EFFECTIVE DESCRIPTIVE SET THEORY

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Contents

Our very eloquent charge from Jan van Mill was to "draw a line to Brouwer" from descriptive set theory, but this proved elusive: in fact there are few references to Brouwer, in Lusin [1928] and Lusin [1930], none of them substantial; and even though Brouwer refers to Borel, Lebesgue and Hadamard in his early papers, it does not appear that he was influenced by their work in any substantive way.[1] We have not found any references by him to more developed work in descriptive set theory, after the critical Lebesgue [1905]. So instead of looking for historical connections

[1] Cf. Michel [2008], who traces the origin and dismisses the significance of terms like "semi-" or "pre-" intuitionists.

or direct influences (in either direction), we decided to identify and analyze some basic themes, problems and results which are common to these two fields; and, as it turns out, the most significant connections are between intuitionistic analysis and *effective* descriptive set theory, hence the title.

⇒ We will outline our approach and (limited) aims in Section 1, marking with an arrow (like this one) those paragraphs which point to specific parts of the article. Suffice it to say here that our main aim is to identify a few, basic results of descriptive set theory which can be formulated and justified using principles that are both intuitionistically and classically acceptable; that we will concentrate on the mathematics of the matter rather than on history or philosophy; and that we will use standard, classical terminology and notation.

This is an elementary, mostly expository paper, broadly aimed at students of logic and set theory who also know the basic facts about recursive functions but need not know a lot about either intuitionism or descriptive set theory. The only (possibly) new result is Theorem 6.1, which justifies simple definitions and proofs by induction in Kleene's Basic System of intuitionistic analysis, and is then used in Theorems 7.1 and 7.2 to give in the same system a rigorous definition of the Borel sets and prove that they are analytic; the formulation and proof of this last result is one example where methods from effective descriptive set theory are used in an essential way.

§1. Introduction. The "founding" documents for our two topics are Brouwer [1907], [1908] for intuitionism and (plausibly) Lebesgue [1905] for descriptive set theory.[2] It was a critical and confusing time for the foundations of mathematics, with the existence of "antinomies" (such as Russell's Paradox[3]) in the naive understanding of Cantor's set theory sinking in and especially after the proof of the Wellordering Theorem by Zermelo [1904]: this was based on the *Axiom of Choice*, which had been routinely used by Cantor (and others) but not formulated before in full generality or placed at the center of a solution to an important open problem.[4] The "foundational crisis" started with these developments was not "resolved" until (at least) the 1930s, after a great deal of work by mathematicians and philosophers which has been richly documented and analyzed and is not (thank God) our topic here. We will only comment briefly in the next four paragraphs on a few issues which are important for what we want to say and how we will try to say it.

1A. Logic and mathematics. There was no clear distinction between logic and mathematics at the turn of the 20th century, especially with the extremes of Frege's *logicism* which took mathematics to be part of logic and Brouwer's view that "logic depends upon mathematics".[5] For example, the eloquent *Five Letters* by Baire, Borel, Hadamard and Lebesgue in Hadamard [1905] do not refer explicitly to the logicist view, but still appear to understand the Axiom of Choice to be primarily a principle of logic: some of the arguments in them seem to consider seriously the possibility of making infinitely many choices of various kinds *in the course of a*

[2]Cf. the introduction to Y. N. Moschovakis [2009] and the excellent Kanamori [2010], which traces the subject a bit further back to the work of Cantor, Borel [1898] and Baire [1899].

[3]cf. Van Heijenoort [1967].

[4]Moore [1982, Chapter 1].

[5]Brouwer [1907, Chapter 3].

proof.[6] Zermelo [1908] probably deserves the credit for first separating logical from mathematical assumptions in set theory, by including the existence of an infinite set among his axioms; compare this with the "proof" by logic alone of the existence of an infinite number sequence in Dedekind [1888].

Classical first-order logic was first separated from mathematical assumptions and formulated precisely in Hilbert [1918], after Brouwer [1908] had already rejected one of its cardinal principles, the *Law of Excluded Middle*

$$(\mathsf{LEM}) \qquad \text{for every proposition } P,\ P \vee \neg P.$$

This radical act was " ... *like denying the astronomer the telescope or the boxer the use of his fists*" according to Hilbert, as Kleene [1952] quotes him.

Intuitionistic propositional logic was formulated precisely in Heyting [1930a], basically as a system in the style of Hilbert with $\neg P \to (P \to Q)$ replacing Hilbert's stronger *Double Negation Elimination* scheme, $\neg\neg P \to P$. *Intuitionistic first-order logic* can be abstracted from Heyting [1930b].[7]

⇒ For our part, *we will assume intuitionistic logic and the mathematical axioms in the Basic System* **B** of Kleene and Vesley [1965] and Kleene [1969], which are intuitionistically acceptable and classically true. *We will summarize these in Section 2*, and we will include among the hypotheses of a claim any additional mathematical or logical hypotheses that we use, including LEM.

⇒ The early researchers in descriptive set theory—Baire, ... , Lebesgue but also Lusin—talked a lot about "constructive proofs", but it was the mathematics they were worrying about, not the logic, and they used LEM with abandon right from the get go. In Section 4E we will point out that starting in 1925, *they used the full strength of* **B** + LEM, a well-known, very strong system in which most of classical analysis can be formalized.

⇒ More recently, there have been serious attempts to develop *intuitionistic descriptive set theory*, most prominently by Veldman [1990], [2008], [2009] (and the many references there) and Aczel [2009], and we will comment briefly on this work.

1B. $\mathbb{N}, \mathbb{R}, \mathcal{N}$ **and the second number class.** Both Brouwer and the early descriptive set theorists accepted whole-heartedly the natural numbers $\mathbb{N}$, albeit with vigorous arguments on whether it should be viewed as a *completed* or a *potential* (infinite) totality. They also accepted the real number field $\mathbb{R}$ and all their early work was about $\mathbb{R}$, $\mathbb{R}^n$ and real-valued functions of several variables; this is true not only of Brouwer [1907] and Lebesgue [1905], but also the crucial, later Suslin [1917] and Lusin [1917].

⇒ The descriptive set theorists also accepted uncritically the *Baire space*

$$\mathcal{N} = \mathbb{N}^{\mathbb{N}} = \text{ the set of all infinite sequences of natural numbers.}$$

[6] Lebesgue to Borel: "To make an infinity of choices cannot be to write down or to name the elements chosen, one by one; life is too short."

[7] Cf. J. R. Moschovakis [2015] for an introduction or Kleene [1952, p. 82ff] for an equivalent system. Classically trained logicians who want to understand the relation between classical and intuitionistic logic may want to look at the system in Gentzen [1934-35], most easily accessible from its exposition in Kleene [1952, Chapter XV].

Moreover, it soon became clear that it is much easier to formulate and prove results about $\mathcal{N}$ which then "transfer" with little additional work to $\mathbb{R}$, and *we will simplify our task considerably by following the classic* Lusin [1930] *in this*.[8]

Brouwer [1918] also accepted the Baire space, which he called the *universal spread*. He identified it with the process of generating its members by choosing freely a natural number, and after each choice again choosing freely a natural number, etc., thus producing the potentially infinite *choice sequences* which are the elements of $\mathcal{N}$. This view of Baire space as a potential totality ultimately provides the justification for *continuity principles* which are at the heart of intuitionistic analysis but classically false.

⇒ We will formulate and discuss briefly the continuity principles in Sections 2A and 9B, but *we will not assume or use any of them*.

Both Brouwer and the descriptive set theorists mistrusted the totality of all (countable) ordinals and worried that proofs by induction or definitions by recursion on Cantor's *Second Number Class* might lead to contradictions or (at least) to results which might not be certain.[9] Some of the blame for this mistrust should fall on Cantor, whose "definition" (or axiomatization) of ordinals in Cantor [1883], [1895], [1897] is rather heavy and includes (in the first of these) a reference to

> … the law that it is always possible to put every well-defined set into the form of a well-ordered set – a law of thought which seems to me to be basic …

Reading this, it is not difficult to suppose that the theory of ordinals is entangled with the Wellordering Principle which these mathematicians did not accept.

⇒ In any case, as we specify it in Section 2, Kleene's **B** *does not refer to or assume anything about ordinals*. Brouwer [1918], Veldman [2008], [2009] and Aczel [2009] develop intuitionistic theories of ordinals, but we will not deal with it here.

⇒ **1C. Constructions vs. definitions.** In constructive mathematics, a proof of $(\exists x)P(x)$ is expected to yield a *construction* of some object x which can be proved to have the property P, whatever "constructions" are—and they are typically taken (explicitly or implicitly) to be primitives, rather like "sets" in classical mathematics. For formalized intuitionistic theories, this $\exists$-*Principle* can be made precise and proved in many ways, including most significantly by using various *realizability notions* in the sense of Kleene. *We will discuss some of these results and their consequences in Section 9A*.

In descriptive set theory, Lebesgue [1905] starts by expressing doubts about the general conception of a function $f : \mathbb{R}^n \to \mathbb{R}$ as an *arbitrary correspondence* (in the sense of Dirichlet and Riemann); he claims that in practice, mathematicians are most interested in functions which are *analytically representable*—by explicit formulas, infinite series, limits and the like; and in the crucial, fourth paragraph of his seminal paper, he argues that

[8] The classical theory was eventually developed for arbitrary *separably and completely metrizable* (Polish) *topological spaces* and several methods have been established for *transferring* results about $\mathcal{N}$ to all of them, cf. Kechris [1995], Y. N. Moschovakis [2009].

[9] Consider the title of Suslin [1917], *On a definition of Borel measurable sets without transfinite numbers* whose main result is (the classical version of) Corollary 8.2 below. Today, we understand that the theory of ordinals does not depend on the Axiom of Choice, set theorists consider definitions by ordinal recursion as constructive, and Suslin might well title his paper *On a definition of analytic-co-analytic sets without quantification over the continuum*.

> ... [if there are real functions which are not analytically representable], it is important to study the common properties of all [those which are].

A limited goal, but it soon grew to a general understanding of the field as the *definability theory of the continuum*.[10]

There were some discussions of what should count as a *definition*, especially in Hadamard [1905] where Baire says in his letter that

> Progress in this matter would consist in delimiting the domain of the definable,

"this matter" being the question of *what mathematical objects exist* (and what this means). It is a difficult matter and not much general progress was made in it; in practice, they studied sets and functions which can be *defined* starting with $\mathbb{N}$ and the *open* subsets of $\mathcal{N}$ (and $\mathbb{R}$) and applying standard set-theoretic operations, including countable unions and intersections, definition by recursion on the countable ordinals (reluctantly), but also complementation and (after 1925) quantification over $\mathcal{N}$, which are dubious as *constructions* of sets.

$\Rightarrow$ The effective theory sidesteps the thorny problem of *what definitions are* by (in effect) axiomatizing the theory of *sets of definable objects*: the members of a *coded set* come with attached "codes" which are assumed to provide definitions of some kind for them. In Section 4A we will explain this idea for the special case where the codings are in $\mathcal{N}$, after we summarize briefly in Section 3 the basic facts about *computability on Baire space*; this allows us to act effectively on $\mathcal{N}$-coded objects and prove intuitionistically many of their properties by formulating them in terms of their codings. It is the key feature of the effective theory, and we will use it systematically in Sections 4 – 8, the main part of this article.

1D. Intuitionistic refinements of classical results. A common methodological practice in constructive mathematics (of all flavors) is to "refine" classical results, using definitions of the relevant notions which are classically equivalent to the standard ones and yield versions that are more suitable to constructive analysis. Sometimes one, specific refinement is deemed to be "the natural constructive version" of the result in question, but it is more common to consider several reformulations with different constructive status—some of them provable and some not. This legitimate and important part of constructive mathematics often bewilders the classical mathematician: she mostly wants to understand the constructive meaning and (possibly) proof of theorems she thinks she understands, and sometimes she cannot even recognize them in their refined versions.

$\Rightarrow$ In any case, we will not do this: with a few exceptions (noted mostly in footnotes), *we will fix just one*, carefully chosen but *standard definition for each of the notions we need*, with the full knowledge that these may be understood differently in classical and in intuitionistic mathematics.

§2. Our assumptions. We will naturally formulate the mathematical axioms we use and our results and comments in the (informal) language used by descriptive set theorists. Here we specify a many-sorted formal system $\mathbf{B}^*$ in which they can

[10]Cf. Y. N. Moschovakis [2009, Introduction] and Kanamori [2010] (which starts with this declaration).

all (in principle) be formalized; briefly, it is a conservative extension of a part of Kleene's *Intuitionistic Analysis* which is classically sound.

2A. Kleene's Basic System B. Kleene formalized intuitionistic analysis in Kleene and Vesley [1965] and Kleene [1969]. He uses a two-sorted, first-order language, with variables $i, j, \dots$ of sort $\mathbb{N}$ and $\alpha, \beta, \dots$ of sort $\mathcal{N}$, naturally varying over $\mathbb{N}$ and $\mathcal{N}$ in the intended interpretation. There are finitely many constants for (primitive recursive) functions with arguments in $\mathbb{N}$ and $\mathcal{N}$ and values in $\mathbb{N}$ and there are terms of both sorts and formation rules $(u, t) \mapsto u(t)$ and $t \mapsto (\lambda j)t$ which create them when u and t are of respective sorts $\mathcal{N}$ and $\mathbb{N}$. Identity is primitive for terms of sort $\mathbb{N}$ and satisfies LEM, $s = t \vee s \neq t$ (with $s \neq t :\equiv \neg s = t$); it is defined for terms of sort $\mathcal{N}$,

$$u = v :\equiv (\forall i)[u(i) = v(i)].$$

The *Basic Fragment* **B** of Kleene's full system **I** comprises the following:[11]

(**B**1) The *standard axioms for arithmetic, with full induction over arbitrary formulas*. This makes it possible to express formally and prove the basic properties of all *primitive recursive* and *recursive* (Turing computable) functions on $\mathbb{N}$. We fix a formula $\mathrm{GR}(\alpha)$ which defines the set of *recursive points* of $\mathcal{N}$, perhaps

$$\text{(2-1)}\qquad \mathrm{GR}(\alpha) :\equiv (\exists e)(\forall i)[(\exists j)T_1(e, i, j)\ \&\ (\forall j)(T_1(e, i, j) \to \alpha(i) = U(j))]$$

with T_1, U from the Normal Form Theorem IX in Kleene [1952].

(**B**2) The *Countable Axiom of Choice* (for $\mathcal{N}$)

$$(\mathrm{AC}^0_1)\qquad (\forall i)(\exists\alpha)R(i, \alpha) \Longrightarrow (\exists\delta)(\forall i)R(i, (\delta)_i).$$

(**B**3) Proof by *Bar Induction*. This is a powerful method of proof in **B** which we will formulate rigorously when we need it, in Section 5B.[12]

These assumptions are intuitionistically acceptable and classically valid, as opposed to the *Continuity Axioms*[13] of the full system **I** which are classically false or

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[11] We assume a standard coding of tuples of natural numbers (a bit different from that in Kleene and Vesley [1965]), an injection $(u_0, \dots, u_{n-1}) \mapsto \langle u_0, \dots, u_{n-1}\rangle \in \mathbb{N}$ of finite sequences from $\mathbb{N}$ for which $\langle\emptyset\rangle = 1$ and for suitable primitive recursive functions and relations

$$c_n(u_0, \dots, u_{n-1}) = \langle u_0, \dots, u_{n-1}\rangle, \quad \mathrm{Seq}(u) \iff (\exists u_0, \dots, u_{n-1})[u = \langle u_0, \dots, u_{n-1}\rangle],$$
$$u \sqsubseteq v \iff \mathrm{Seq}(u)\ \&\ \mathrm{Seq}(v)\ \&\ u \text{ codes an initial segment of } v,$$
$$\mathrm{lh}(\langle u_0, \dots, u_{n-1}\rangle) = n, \quad (\langle u_0, \dots, u_{n-1}\rangle)_i = u_i \text{ (for } i < n, = 0 \text{ otherwise)}$$
$$\langle u_0, \dots, u_{n-1}\rangle * \langle v_0, \dots, v_{m-1}\rangle = \langle u_0, \dots, u_{n-1}, v_0, \dots, v_{m-1}\rangle \quad \text{(concatenation)}.$$

We code sequences from $\mathcal{N}$ of (finite or infinite) length $n \leq \infty$ by setting

$$\langle\langle \alpha_i \mid i < n\rangle\rangle = \langle\langle \alpha_0, \alpha_1, \dots\rangle\rangle = \lambda s \begin{cases} \alpha_i(t) & \text{if } s = \langle i, t\rangle \text{ for some (uniquely determined) } i < n, t, \\ 0, & \text{otherwise.} \end{cases}$$

We also set for each $\alpha \in \mathcal{N}$ and $i \in \mathbb{N}$, $(\alpha)_i = \lambda t\alpha(\langle i, t\rangle)$, so that if $\alpha = \langle\langle \alpha_i \mid i < n\rangle\rangle$, then for $i < n$, $(\alpha)_i = \alpha_i$; for example, with $n = 2$, $(\langle\langle \alpha, \beta\rangle\rangle)_0 = \alpha$ and $(\langle\langle \alpha, \beta\rangle\rangle)_1 = \beta$.

Finally $(i)\hat{}\,\alpha = (i, \alpha(0), \alpha(1), \dots)$ and $\alpha^* = \lambda t\alpha(t + 1)$, so that $(\alpha(0))\hat{}\,\alpha^* = \alpha$.

[12] The formalized version of **B** comprises Postulate Groups A – D and $^{\mathrm{x}}$26.3b (or $^{\mathrm{x}}$26.8) from Kleene and Vesley [1965, pages 13 – 55, 63] and $^{\mathrm{x}}$30.1, $^{\mathrm{x}}$31.1, $^{\mathrm{x}}$31.2 in Kleene [1969].

[13] Kleene and Vesley [1965, $^{\mathrm{x}}$27.1] and its consequences, *27.2 and *27.15. These are justified by appealing to Brouwer's conception of $\mathcal{N}$ discussed in Section 1B and the weakest of them claims that *every function* $f : \mathcal{N} \to \mathbb{N}$ *is continuous*. We will specify the full system **I** and discuss it briefly in Section 9B.

the (general) *Law of Excluded Middle* LEM which is not intuitionistically acceptable. When we need additional hypotheses, we will "decorate" our claims with them, e.g.,

THEOREM (LEM) ... means that the proof of ... is classical.

Markov's Principle. In a couple of crucial places we will assume in this way

$$\text{(MP)} \qquad (\forall\alpha)\Big(\neg(\forall i)[\alpha(i)=0] \Longrightarrow (\exists i)[\alpha(i)\neq 0]\Big).$$

This is certainly true classically and it is a fundamental assumption of the Russian school of *constructive* (or *recursive*) analysis initiated by Markov [1954]. It is rejected, or at least viewed with suspicion by intuitionists and it is neither provable nor refutable in the full system **I**.[14] We will discuss it briefly in Section 9, but we should stress here that *we do not include* MP *in our assumptions*.

The classical extension $\mathbf{B}+\mathsf{LEM}$ $(=\mathbf{B}1+\mathbf{B}2+\mathsf{LEM})$ of **B** is a familiar system, often called *Analysis* or *Second Order Arithmetic with the Countable Axiom of Choice* (AC^0_1), because of its intended interpretation on the universes $\mathbb{N}$ and $\mathcal{N}$. It is well-known that a great deal of classical mathematics can be formalized in $\mathbf{B}+\mathsf{LEM}$, and this includes all of descriptive set theory, at least until the 1960s; so it is only for convenience that we will move to a conservative extension of it which is more expressive.

2B. Product spaces and pointsets; the system $\mathbf{B}^*$. To study relations and functions with arguments and values in $\mathbb{N}$ and $\mathcal{N}$, we need to consider *product spaces* of the form

$$\text{(2-2)} \qquad X = X_1 \times \cdots \times X_n \text{ with each } X_i = \mathbb{N} \text{ or } \mathcal{N};$$

by convention, $X = X_1$ if the *dimension* n is 1,

$$(X_1 \times \cdots \times X_n) \times (Y_1 \times \cdots \times Y_m) = X_1 \times \cdots \times X_n \times Y_1 \times \cdots \times Y_m,$$

and similarly with pairs of points:

if $x = (x_1,\dots,x_n)$ and $y = (y_1,\dots,y_m)$, then $(x,y) = (x_1,\dots,x_n,y_1,\dots,y_m)$.

For each product X, we add to the language of **B** *pointset variables* $A, B, \dots$ *of sort* X which vary over subsets of X and the corresponding prime formulas

$$(s_1,\dots,s_n) \in A \text{ or, synonymously } A(s_1,\dots,s_n)$$

where each s_i is a term of sort $\mathbb{N}$ or $\mathcal{N}$ as required. We will also abbreviate

$$\text{(2-3)} \qquad x \in A^c :\equiv x \notin A :\equiv \neg A(x) \quad \text{(complementation)}.$$

The formal system $\mathbf{B}^*$ in which all we will do can be routinely formalized is obtained by allowing *formulas with no bound set variables* in clauses ($\mathbf{B}1$) – ($\mathbf{B}3$) of the description of **B** above and adding the following:

($\mathbf{B}4$) *Congruence Axioms*

$$s_1 = t_1 \;\&\; \cdots \;\&\; s_n = t_n \Longrightarrow \Big(A(s_1,\dots,s_n) \leftrightarrow A(t_1,\dots,t_n)\Big).$$

($\mathbf{B}5$) *Comprehension Axioms*

$$(\forall\vec{v})(\exists A)(\forall\vec{u})[A(\vec{u}) \iff \phi(\vec{u},\vec{v})],$$

[14] Kleene and Vesley [1965, pages 129 – 131, 186], see Sections 9A and 9C, below.

one for each *formula $\phi(\vec{u},\vec{v})$ with the indicated free variables and no bound pointset variables*, subject to the obvious additional formal restrictions.

The classical system $\mathbf{B}^* + \mathsf{LEM}$ is related to $\mathbf{B} + \mathsf{LEM}$ exactly as Gödel-Bernays set theory is related to ZF (or ZFC) and by the usual (classical) proof it is *conservative over* $\mathbf{B} + \mathsf{LEM}$—because every (two-sorted) model $\mathcal{M}$ of the latter can be extended to a (many-sorted) model in which the fresh set variables are interpreted by the sets definable (with parameters) in $\mathcal{M}$.

From the intuitionistic point of view, the pointset variables of sort X range over *mathematical species* or *extensional properties* of elements of X which are legitimate mathematical entities by the "Second Act of Intuitionism", cf. Brouwer [1918], [1952]. A (classical) modification of the argument above using *Kripke models* can be used to prove that $\mathbf{B}^*$ *is conservative over* $\mathbf{B}$ and $\mathbf{B}^* + \mathsf{MP}$ *is conservative over* $\mathbf{B} + \mathsf{MP}$.

2C. The course-of-values function. For $x = (x_1, \dots, x_n) \in X$ as in (2-2), set $x_i(t) = x_i$ if $X_i = \mathbb{N}$, so that $x_i(t)$ makes sense for $i = 1, \dots, n$, and define the function $(x, t) \mapsto \overline{x}(t)$ by the (primitive) recursion

$$\overline{x}(0) = 1, \quad \overline{x}(t+1) = \overline{x}(t) * \langle x_1(t), \dots, x_n(t)\rangle, \tag{2-4}$$

so that for $t > 0$,

$$\overline{x}(t) = \langle x_1(0), \dots, x_n(0), x_1(1), \dots, x_n(1), \dots, x_1(t-1), \dots, x_n(t-1)\rangle.$$

If $X = \mathcal{N}$, then $\overline{\alpha}(t) = \langle \alpha(0), \dots, \alpha(t-1)\rangle$ in the familiar Kleene notation.

We view each X in (2-2) as a topological product of copies of $\mathbb{N}$ (taken discrete) and $\mathcal{N}$ (with its usual topology), and for each $u \in \mathbb{N}$ we set

$$N_u = N_u^X = \{x \in X \mid (\exists t)[u = \overline{x}(t)]\} = \{x \in X \mid (\exists t < u)[u = \overline{x}(t)]\}; \tag{2-5}$$

now each N_u is a clopen set (empty if u is not a sequence code of the proper kind) and these sets form a countable basis for the topology of X.

§3. (Partial) continuity and recursion.[15] A *partial function* $f : X \rightharpoonup W$ is a subset of $X \times W$ which is the graph of a function, i.e.,

$$f : X \rightharpoonup W \iff_{\mathrm{df}} f \subseteq X \times W \;\&\; (\forall x)(\forall w)(\forall w')\Big([f(x,w) \;\&\; f(x,w')] \to w = w'\Big).$$

We use standard notation for these objects:[16]

$$f(x) = w \iff_{\mathrm{df}} f(x,w), \quad f(x)\!\downarrow \iff_{\mathrm{df}} (\exists w)[f(x) = w], \quad f(x)\!\uparrow \iff_{\mathrm{df}} \neg f(x)\!\downarrow,$$
$$D_f = \{x \in X \mid f(x)\!\downarrow\} \quad (\text{the } \textit{domain of convergence of } f).$$

Identities between partial functions are understood *strictly* (strongly), as equalities both in and out of their domains of convergence:

$$(\forall x)[f(x) = g(x)] \iff_{\mathrm{df}} (\forall x, w)[f(x) = w \iff g(x) = w].$$

[15] Many readers will want to skim through the elementary material in this Section, which we have included to fix notions, set notation and "certify" that it can be developed in $\mathbf{B}^*$.

[16] In classical recursion theory on the natural numbers, it is traditional to write $f(x) \simeq w$ rather than $f(x) = w$ when f is partial, but this notation is never used in descriptive set theory partly because it does not work well when $W = \mathcal{N}$.

Most often we will assume or prove equality under hypotheses, in the form

$$x \in A \Longrightarrow f(x) = g(x),$$

which means that

$$x \in A \Longrightarrow (\forall w)[f(x) = w \iff g(x) = w].$$

A partial function $f : X \rightharpoonup \mathbb{N}$ is ***continuous with code*** $\varepsilon \in \mathcal{N}$, if

$$\text{(3-1)} \quad f(x)\downarrow \Longrightarrow \Big(f(x) = w \iff (\exists t)[(\forall i < t)[\varepsilon(\overline{x}(i)) = 0]\ \&\ \varepsilon(\overline{x}(t)) = w + 1]\Big);$$

$g : X \rightharpoonup \mathcal{N}$ is ***continuous with code*** ε if there is a continuous $f : X \times \mathbb{N} \rightharpoonup \mathbb{N}$ with code ε such that

$$\text{(3-2)} \qquad g(x) = \lambda i f(x, i) \text{ with } g(x)\downarrow \iff (\forall i)[f(x, i)\downarrow];$$

and $h : X \rightharpoonup Y_1 \times \cdots \times Y_k$ is ***continuous with code*** ε if $h(x) = (h_1(x), \dots, h_k(x))$ with each $h_i : X \to Y_i$ continuous with code $(\varepsilon)_i$.[17]

A pointset $A \subseteq X$ is ***clopen with code*** ε if its *characteristic function*

$$\text{(3-3)} \qquad \chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise} \end{cases} \qquad (x \in X)$$

is continuous with code ε.

A partial function $f : X \rightharpoonup W$ is ***recursive*** if it is continuous with a recursive (Turing computable) code; a pointset $A \subseteq X$ is ***recursive*** if it is clopen with a recursive code; and a point $w = (w_1, \dots, w_n)$ is ***recursive*** if for each i, either $w_i \in \mathbb{N}$ or $w_i \in \mathcal{N}$ is recursive as a function $w_i : \mathbb{N} \to \mathbb{N}$. It is easy to check that

$$\alpha : \mathbb{N} \to \mathbb{N} \text{ is recursive } \iff \mathrm{GR}(\alpha) \text{ by (2-1).}$$

In the sequel we will talk of "continuous" or "recursive" partial functions and pointsets, skipping explicit reference to the codings unless it is needed.

THEOREM 3.1. (1) *The* (total) *evaluation* $(\alpha, t) \mapsto \alpha(t)$ *and course-of-values maps* $(x, t) \mapsto \overline{x}(t)$ *are recursive.*

(2) *The class of continuous partial functions contains all constants and projections* $(x_1, \dots, x_k) \mapsto x_i$ *and it is closed under*

substitution $\quad f(x) = g(h_1(x), \dots, h_k(x));$

branching $\quad f(i, x) = \text{if } (i = 0) \text{ then } h_1(x) \text{ else } h_2(i, x);$

primitive recursion $\quad f(0, x) = g(x), \quad f(t + 1, x) = h(f(t, x), t, x);$ *and*

λ-substitutions $\quad f(x, y) = h_1(\lambda i g(x, i), y), \quad f(x, y) = h_2(\langle\langle g(x, i) \mid i < \infty\rangle\rangle, y);$

and similarly for the class of recursive partial functions, except that a constant $f(x) = \alpha$ *is recursive only if* α *is recursive.*

(3) *If* $f : X \rightharpoonup W$ *is recursive,* x *is a recursive point of* X *and* $f(x)\downarrow$, *then* $f(x)$ *is a recursive point of* W.

[17]With the usual, topological definition of continuity, classically every $f : X_0 \to W$ which is defined and continuous on some subspace $X_0 \subseteq X$ has a code as a continuous partial function on X. This cannot be proved in $\mathbf{B}^*$, cf. Corollary 9.2.

PROOF. The idea is that a modulus of continuity for the function defined (by substitution etc.) can be computed from moduli of continuity for the given partial functions and we will skip the messy computations needed to implement it. We only note that we understand compositions *strictly*, so that in (2), e.g.,

$$\begin{aligned}&g(h_1(x),\dots,h_k(x)) = w\\ &\iff (\exists w_1,\dots,w_k)[h_1(x)=w_1 \ \&\ \cdots\ \&\ h_k(x)=w_k \ \&\ g(w_1,\dots,w_k)=w],\end{aligned}$$

and that (3) is important for the effective theory. ⊣

Especially useful are the *total recursive functions* which we use to prove *uniform* versions of results, most often by appealing to the following group of definitions and facts sometimes collectively called *the Kleene Calculus* for partial recursion:

THEOREM 3.2. *For any space X, set*

$$\begin{aligned}\{\varepsilon\}^{X,0}(x)=w &\iff (\exists t)[(\forall i<t)[\varepsilon(\overline{x}(i))=0]\ \&\ \varepsilon(\overline{x}(t))=w+1],\\ \{\varepsilon\}^{X,1}(x)=\beta &\iff (\forall i)[\{\varepsilon\}^{X\times\mathbb{N},0}(x,i)=\beta(i)].\end{aligned}$$

Then:

(1) *The partial functions $(\varepsilon,x)\mapsto\{\varepsilon\}^{X,0}(x)$ and $(\varepsilon,x)\mapsto\{\varepsilon\}^{X,1}(x)$* (into $\mathbb{N}$ and $\mathcal{N}$) *are recursive.*

(2) *A partial function $f : X \rightharpoonup \mathbb{N}$ is continuous with code ε if and only if*

$$f(x)\downarrow \implies \Big(f(x)=\{\varepsilon\}^{X,0}(x)\Big);$$

and $g : X \rightharpoonup \mathcal{N}$ is continuous with code ε if and only if

$$g(x)\downarrow \implies \Big(g(x)=\{\varepsilon\}^{X,1}(x)\Big).$$

(3), *The S-Theorem*: *For any two spaces X and Y, there is a* (total) *recursive function $S = S^Y_X : \mathcal{N}\times Y \to \mathcal{N}$ such that*

$$\{\varepsilon\}^{Y\times X,0}(y,x)=\{S(\varepsilon,y)\}^{X,0}(x), \tag{3-4}$$

and similarly with 1 *in place of* 0 *throughout*.

(4), *The* 2*nd Recursion Theorem*: *For every continuous $f : \mathcal{N}\times X \rightharpoonup W$* (with $W=\mathbb{N}$ or $\mathcal{N}$), *there is an $\widetilde{\varepsilon}\in\mathcal{N}$* (*which can be computed from a code of f*) *such that*

$$f(\widetilde{\varepsilon},x)\downarrow \implies f(\widetilde{\varepsilon},x)=\{\widetilde{\varepsilon}\}(x). \tag{3-5}$$

PROOF. (1) and (2) are very easy, from the definitions with a bit of computation.

To prove (3) for $i=0$, with $X=X_1\times\cdots\times X_n$ and $Y=Y_1\times\cdots\times Y_m$, it suffices to define $S(\varepsilon,y)$ so that for all x,y and i,

$$S(\varepsilon,y)(\overline{x}(i))=\varepsilon(\overline{(y,x)}(i)). \tag{3-6}$$

Define first $f:\mathbb{N}\times Y\times\mathbb{N}\to\mathbb{N}$ by the following primitive recursion:

$$\begin{aligned}f(0,y,u)&=1,\\ f(i+1,y,u)&=f(i,y,u)*\langle y_1(i),\dots,y_m(i),(u)_{ni},(u)_{ni+1},\dots,(u)_{ni+n-1}\rangle.\end{aligned}$$

We now claim that for all x,y,i and $t\geq i$,

$$f(i,y,\overline{x}(t))=\overline{(y,x)}(i); \tag{3-7}$$

this is true at the base (when both values are 1) and follows in the induction step directly from the definition (2-4) of the course-of-values function:

$$\begin{aligned} f(i+1, y, \overline{x}(t)) &= f(i, y, \overline{x}(t)) * \langle y_1(i), \dots, y_m(i), (\overline{x}(t))_{ni}, \dots, (\overline{x}(t))_{ni+n-1}\rangle \\ &= f(i, y, \overline{x}(t)) * \langle y_1(i), \dots, y_m(i), x_1(i), \dots, x_n(i)\rangle = \overline{(y, x)}(i+1). \end{aligned}$$

Finally, (3-7) implies (3-6) with $S(\varepsilon, y) = \lambda u \varepsilon(f(\max\{i \mid ni \leq \mathrm{lh}(u)\}, y, u))$.

For the 2nd Recursion Theorem (4), we use Kleene's classical (if opaque) proof: let $S = S_X^{\mathcal{N}} : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ be the function given by (3) with $Y = \mathcal{N}$, and choose by (2) an ε_0 such that

$$f(S(\alpha, \alpha), x)\downarrow \implies \Big(f(S(\alpha, \alpha), x) = \{\varepsilon_0\}(\alpha, x)\Big),$$

which implies that

$$f(S(\varepsilon_0, \varepsilon_0), x)\downarrow \implies \Big(f(S(\varepsilon_0, \varepsilon_0), x) = \{\varepsilon_0\}(\varepsilon_0, x) = \{S(\varepsilon_0, \varepsilon_0)\}(x)\Big)$$

and yields (3-5) with $\widetilde{\varepsilon} = S(\varepsilon_0, \varepsilon_0)$. ⊣

The 2nd Recursion Theorem is a crucial tool of the effective theory, as we will see further down.

The S-Theorem is the natural extension to partial functions with arguments in $\mathbb{N}$ and $\mathcal{N}$ of the familiar S_n^m-Theorem of Kleene in ordinary recursion theory and is used in the same way here, to produce *uniform versions of results from constructive proofs*. We will define rigorously what this means in Section 4A, but the following uniform version of (2) in Theorem 3.1 is a typical example.

LEMMA 3.3 (Uniformity of substitution). *The class of continuous partial functions is **uniformly closed** under substitution*; *i.e., for every space X and any $k \geq 1$, there is a total recursive function $\boldsymbol{u} : \mathcal{N}^{k+1} \to \mathcal{N}$ such that if $\widetilde{g}, \widetilde{h}_1, \dots, \widetilde{h}_k$ are codes of continuous partial functions $g, h_1, \dots, h_k$* (of the proper sorts) *and*

$$f(x) = g(h_1(x), \dots, h_k(x)),$$

then f is continuous with code $\boldsymbol{u}(\widetilde{g}, \widetilde{h}_1, \dots, \widetilde{h}_k)$.

PROOF. The partial function

$$\varphi(\alpha, \beta_1, \dots, \beta_k, x) = \{\alpha\}(\{\beta_1\}(x), \dots, \{\beta_k\}(x))$$

is recursive by (1) of Theorem 3.2 and (2) of Theorem 3.1; choose a recursive code $\widetilde{\varphi}$ of it and set

$$\boldsymbol{u}(\widetilde{g}, \widetilde{h}_1, \dots, \widetilde{h}_k) = S(\widetilde{\varphi}, \widetilde{g}, \widetilde{h}_1, \dots, \widetilde{h}_k)$$

with the relevant S. ⊣

§4. The basic coded pointclasses. We introduce here the simplest ***pointclasses***—collections of pointsets—of classical descriptive set theory, with some care, so we can prove a good deal about these objects using only our assumptions. The key is to pair the members of a pointclass $\mathbf{\Lambda}$ with *codes* (in $\mathcal{N}$), as we did for continuous partial functions: intuitively, a $\mathbf{\Lambda}$-code for some $P \subseteq X$ specifies a "definition" of P which puts it in $\mathbf{\Lambda}$.

4A. Coded sets and uniformities. In broadest generality, a ***coded set*** is a set $\mathbf{A}$ together with a *coding map*, a surjection

$$c^{\mathbf{A}} : C^{\mathbf{A}} \twoheadrightarrow \mathbf{A}$$

of the *set of codes* $C^{\mathbf{A}} \subseteq \mathcal{N}$, onto $\mathbf{A}$; the *lightface* or ***effective part*** A of $\mathbf{A}$ comprises its members which have recursive codes,

$$\mathrm{A} = \{c^{\mathbf{A}}(\alpha) \mid \alpha \in C^{\mathbf{A}} \;\&\; \alpha \text{ is recursive}\};$$

and for any two coded sets $\mathbf{A}, \mathbf{B}$, a $\forall$-$\exists$ proposition

$$(\forall P \in \mathbf{A})(\exists Q \in \mathbf{B})R(P, Q) \tag{4-1}$$

holds uniformly, if there is a recursive partial function $\boldsymbol{u} : \mathcal{N} \rightharpoonup \mathcal{N}$ such that

$$\alpha \in C^{\mathbf{A}} \Longrightarrow \Big(\boldsymbol{u}(\alpha)\!\downarrow \;\&\; \boldsymbol{u}(\alpha) \in C^{\mathbf{B}} \;\&\; R(c^{\mathbf{A}}(\alpha), c^{\mathbf{B}}(\boldsymbol{u}(\alpha)))\Big) \quad (\alpha \in \mathcal{N}). \tag{4-2}$$

The definition extends trivially to $\vec{\forall}$-$\exists$ propositions

$$(\forall \vec{P} \in \vec{\mathbf{A}})(\exists Q \in \mathbf{B})R(\vec{P}, Q)$$

on tuples from coded sets as in Lemma 3.3 where, in fact, the ***uniformity u*** is total; this is often the case and it simplifies matters, but it is not essential.

We have included the full, pedantic definition of coded sets and uniform truth for the sake of completeness but in practice, we will define coded sets in the form

$$P \text{ is (in) } \mathbf{A} \text{ with code } \alpha \iff \cdots$$

without explicitly introducing a name for the coding map $c^{\mathbf{A}} : C^{\mathbf{A}} \twoheadrightarrow \mathbf{A}$ or (in some cases) ever mentioning it again. It is easier—and better—to understand these notions intuitively from the several examples we will give, in some cases spelling out exactly what the uniformities achieve.[18]

Codings and uniform truth are most important for the effective theory, but the very existence of a code of a certain kind for an object P sometimes has important implications in the intuitionistic theory. Consider the following simplest example where a *strong code* for a set $P \subseteq \mathbb{N}$ is any $\chi \in \mathcal{N}$ such that

$$n \in P \iff \chi(n) = 1.$$

LEMMA 4.1. *A set $P \subseteq \mathbb{N}$ has a strong code if and only if its membership relation satisfies* LEM, *in symbols,*[19]

$$(\exists\alpha)(\forall n)[n \in P \iff \alpha(n) = 1] \iff (\forall n)[n \in P \vee n \notin P]. \tag{$*$}$$

PROOF. The direction $(\Rightarrow)$ of $(*)$ holds because

$$\mathbf{B} \vdash (\forall\alpha)(\forall n)[\alpha(n) = 1 \vee \alpha(n) \neq 1],$$

and for the direction $(\Leftarrow)$ we use

$$\mathbf{B} \vdash P \vee Q \leftrightarrow (\exists i)[(i = 1 \to P) \;\&\; (i \neq 1 \to Q)],$$

[18] Coded sets and uniformities are discussed in considerable detail in Y. N. Moschovakis [2010a], where they are put to heavier use than we need to put them here.

[19] In intuitionistic mathematics, a relation $P \subseteq X$ which satisfies $(\forall x)[P(x) \vee \neg P(x)]$ is called *decidable*. We will not use this terminology, to avoid confusion with the standard, classical identification of “decidable” with “recursive”.

as well as $(\forall n)(\exists i)R(n,i) \Longrightarrow (\exists \alpha)(\forall n)R(n,\alpha(n))$, which is an easy consequence of the Countable Axiom of Choice. ⊣

4B. Open and closed pointsets. For any product space X, a subset $G \subseteq X$ is ***open*** (or $\underset{\sim}{\mathbf{\Sigma}}^0_1$) ***with code*** α, if [20]

$$x \in G \iff \{\alpha\}^{X,0}(x)\downarrow \iff (\exists t)[\alpha(\overline{x}(t)) > 0]; \tag{4-3}$$

and a subset $F \subseteq X$ is ***closed*** (or $\underset{\sim}{\mathbf{\Pi}}^0_1$) ***with code*** α if

$$x \in F \iff \{\alpha\}^{X,0}(x)\uparrow \iff (\forall t)[\alpha(\overline{x}(t)) = 0]. \tag{4-4}$$

A pointset $G \subseteq X$ is ***recursively open*** (or Σ^0_1) if it has a recursive $\underset{\sim}{\mathbf{\Sigma}}^0_1$-code and ***recursively closed*** (or Π^0_1) if it has a recursive $\underset{\sim}{\mathbf{\Pi}}^0_1$-code.

Lemma 4.2. *The pointclasses $\underset{\sim}{\mathbf{\Sigma}}^0_1$ and $\underset{\sim}{\mathbf{\Pi}}^0_1$ are uniformly closed under total continuous substitutions; i.e., for the first claim, for all X and $Y = Y_1 \times \cdots \times Y_m$, there is a total, recursive $\boldsymbol{u} : \mathcal{N}^{1+m} \to \mathcal{N}$ such that if $Q \subseteq Y$ is $\underset{\sim}{\mathbf{\Sigma}}^0_1$ with code α_Q, each $h_i : X \to Y_i$ is continuous with code ε_i for $i = 1, \dots, m$, and*

$$P(x) \iff Q(h_1(x), \dots, h_m(x)), \tag{4-5}$$

then P is $\underset{\sim}{\mathbf{\Sigma}}^0_1$ with code $\boldsymbol{u}(\alpha_Q, \varepsilon_1, \dots, \varepsilon_m)$.

Proof. By Lemma 3.3, there is a recursive $\boldsymbol{u}$ such that the map

$$x \mapsto \{\alpha_Q\}(h_1(x), \dots, h_m(x))$$

is continuous with code $\alpha_P = \boldsymbol{u}(\alpha_Q, \varepsilon_1, \dots, \varepsilon_m)$, and then this α_P is a $\underset{\sim}{\mathbf{\Sigma}}^0_1$-code of P, by the definition. ⊣

Lemma 4.3. *For each X, there is a total recursive $f^X = f : \mathcal{N} \times X \times \mathbb{N} \to \mathbb{N}$ such that for all $G, F \subseteq X$ and α,*

$$\text{if } G \text{ is in } \underset{\sim}{\mathbf{\Sigma}}^0_1 \text{ with code } \alpha, \text{ then } \Big(x \in G \iff (\exists t)[f(\alpha, x, t) = 0]\Big), \text{ and} \tag{4-6}$$

$$\text{if } F \text{ is in } \underset{\sim}{\mathbf{\Pi}}^0_1 \text{ with code } \alpha, \text{ then } \Big(x \in F \iff (\forall t)[f(\alpha, x, t) \neq 0]\Big). \tag{4-7}$$

It follows that a set $G \subseteq X$ is $\underset{\sim}{\mathbf{\Sigma}}^0_1$ (Σ^0_1) if and only if there is a continuous (recursive) $g : X \times \mathbb{N} \to \mathbb{N}$ such that

$$x \in G \iff (\exists t)[g(x,t) = 0]; \tag{4-8}$$

and a set $F \subseteq X$ is $\underset{\sim}{\mathbf{\Pi}}^0_1$ (Π^0_1) if and only if

$$x \in F \iff (\forall t)[g(x,t) \neq 0] \tag{4-9}$$

with some continuous (recursive) $g : X \times \mathbb{N} \to \mathbb{N}$.

Proof. Set $f^X(\alpha, x, t) = \text{if } ((\forall i < t)[\alpha(\overline{x}(i)) = 0]) \text{ then } 1 \text{ else } 0$. ⊣

We say that $P \subseteq \mathbb{N} \times X$ is defined by ***branching*** from $Q \subseteq X$ and $R \subseteq \mathbb{N} \times X$ if

$$\begin{aligned} P(i,x) &\iff \text{if } (i = 0) \text{ then } Q(x) \text{ else } R(i,x) \\ &\iff [i = 0 \;\&\; Q(x)] \vee [i \neq 0 \;\&\; R(i,x)] \\ &\iff [i = 0 \to Q(x)] \;\&\; [i \neq 0 \to R(i,x)]. \end{aligned} \tag{4-10}$$

[20] Classically, every open pointset has a $\underset{\sim}{\mathbf{\Sigma}}^0_1$-code, but this cannot be proved in $\mathbf{B}^*$, see Corollary 9.2.

THEOREM 4.4. (1). *$\underset{\sim}{\mathbf{\Sigma}}^0_1$, Σ^0_1, $\underset{\sim}{\mathbf{\Pi}}^0_1$ and Π^0_1 are all closed under conjunction* &, *bounded universal number quantification $\forall^{\leq}$ and branching*;

• *$\underset{\sim}{\mathbf{\Sigma}}^0_1$ and Σ^0_1 are also closed under disjunction $\vee$ and existential number quantification $\exists^{\mathbb{N}}$; and*

• *$\underset{\sim}{\mathbf{\Pi}}^0_1$ and Π^0_1 are also closed under universal number quantification.*

(2) *If $G \subseteq X$ is open with code α, then its complement $G^c = \{x \in X \mid x \notin G\}$ is closed with the same code.*

(3) (MP) *If $F \subseteq X$ is closed with code α, then its complement F^c is open with the same code.*

(4) (LEM). *Both $\underset{\sim}{\mathbf{\Sigma}}^0_1$ and $\underset{\sim}{\mathbf{\Pi}}^0_1$ are closed under disjunction $\vee$, conjunction & and bounded quantification of both kinds, $\exists^{\leq}, \forall^{\leq}$.*

PROOF. (1) We put down some of the many equivalences that need to be checked (from our assumptions) to verify these closure properties:

$$(\exists t)[f(x,t)=0] \,\&\, (\exists s)[g(x,s)=0] \iff (\exists u)[\max(f(x,(u)_0), g(x,(u)_1)) = 0];$$

$$\begin{aligned}[][i = 0 \,\&\, (\exists t)(f(x,t)=0)] \vee [i \neq 0 \,\&\, (\exists s)(g(x,s)=0)] &\iff (\exists t)[(i=0 \,\&\, f(x,t)=0) \vee (i \neq 0 \,\&\, g(x,t)=0)] \\ &\iff (\exists t)[(i + f(x,t)) \cdot ((1 \dot{-} i) + g(x,t)) = 0]];\end{aligned}$$

$$(\exists t)[f(x,t)=0] \vee (\exists s)[g(x,s)=0] \iff (\exists u)[\min(f(x,u), g(x,u)) = 0];$$

$$(\exists s)(\exists t)[f(x,s,t)=0] \iff (\exists u)[f(x,(u)_0,(u)_1) = 0];$$

$$\begin{aligned}\text{if } (i=0) \text{ then } (\exists s)P(s) \text{ else } (\exists s)Q(i,s) &\iff (\exists s)[(i=0 \,\&\, P(s)) \vee (i \neq 0 \,\&\, Q(i,s))] \\ &\iff (\exists s)[\text{if } (i=0) \text{ then } P(s) \text{ else } Q(i,s)].\end{aligned}$$

(2) We use the following equivalences which are provable from our assumptions:

$$\neg(\exists t)[\alpha(\overline{x}(t)) > 0] \iff (\forall t)\neg[\alpha(\overline{x}(t)) > 0] \iff (\forall t)[\alpha(\overline{x}(t)) = 0].$$

(3) If for all x, $x \in F \iff (\forall t)[\alpha(\overline{x}(t)) = 0]$, then

$$x \notin F \iff \neg(\forall t)[\neg\alpha(\overline{x}(t)) > 0] \iff (\exists t)[\alpha(\overline{x}(t)) > 0],$$

using MP and the fact that $\neg\alpha(n) > 0 \iff \alpha(n) = 0$. ⊣

The next definition and result are the keys to proving that all these closure properties of $\underset{\sim}{\mathbf{\Sigma}}^0_1$ and $\underset{\sim}{\mathbf{\Pi}}^0_1$ hold uniformly:

4C. Good universal sets. Fix a coded pointclass $\underset{\sim}{\mathbf{\Gamma}}$ which is uniformly closed under (total) continuous substitutions as in (4-5).[21]

A pointset $U \subseteq \mathcal{N} \times X$ is a (good) ***universal set for $\underset{\sim}{\mathbf{\Gamma}}$ at a space** X*, if

(U1) U is in Γ, the lightface part of $\underset{\sim}{\mathbf{\Gamma}}$;

[21] The use of boldface and lightface fonts to name a coded pointclass and its lightface part has been standard since the 1950s, but the important distinction between $\mathbf{\Gamma}$ and Γ is not easy to read in some fonts, so it has also become standard to use $\underset{\sim}{\mathbf{\Gamma}}$ and Γ instead.

(U2) every pointset $P \subseteq X$ in $\underset{\sim}{\Gamma}$ is a section of U, i.e., for some α,

$$P = U_\alpha = \{x \in X \mid U(\alpha, x)\}; \text{ and} \tag{4-11}$$

(U3) for every Y and every $Q \subseteq Y \times X$ in Γ, there is a recursive $S^Q : Y \to \mathcal{N}$ such that $Q(y, x) \iff U(S^Q(y), x) \quad (x \in X, y \in Y)$.

If (4-11) holds, we call α *a code of P in the coding of $\underset{\sim}{\Gamma}$ induced by U*, and (U3) implies easily that *P is in Γ exactly when it has a recursive code* in this coding.

THEOREM 4.5. *For every X, the set*

$$G_1^{X,0}(\alpha, x) \iff \{\alpha\}(x)\downarrow$$

is universal for $\underset{\sim}{\mathbf{\Sigma}}^0_1$ at X and induces the standard coding for $\underset{\sim}{\mathbf{\Sigma}}^0_1$; and the set

$$F_1^{X,0}(\alpha, x) \iff \{\alpha\}(x)\uparrow$$

is universal for $\underset{\sim}{\mathbf{\Pi}}^0_1$ at X and induces its standard coding.

PROOF. With $U = G_1^{X,0}$, (U1) and (U2) are immediate, by the coding we chose and the closure properties of Σ^0_1. To prove (U3) for a given $Q \subseteq Y \times X$ in Σ^0_1, choose a recursive code ε_Q for it so that

$$Q(y, x) \iff \{\varepsilon_Q\}(y, x)\downarrow \iff \{S(\varepsilon_Q, y)\}(x)\downarrow$$

with the recursive $S : \mathcal{N} \times Y \to \mathcal{N}$ of (3) of Theorem 3.2 and set $S^Q(y) = S(\varepsilon_Q, y)$.⊣

COROLLARY 4.6. *The closure properties of $\underset{\sim}{\mathbf{\Sigma}}^0_1$ all hold uniformly: for example, there is a recursive function $\boldsymbol{u}(\alpha_Q, \beta_R)$ such that if*

$$P(x) \iff (\exists t)[Q(x, t) \;\&\; R(t)]$$

and α_Q, β_R are $\underset{\sim}{\mathbf{\Sigma}}^0_1$-codes of $Q \subseteq X \times \mathbb{N}$ and $R \subseteq \mathbb{N}$, then $\boldsymbol{u}(\alpha_Q, \beta_R)$ is a $\underset{\sim}{\mathbf{\Sigma}}^0_1$-code of P.

PROOF. The relation

$$P^*(\alpha, \beta, x) \iff (\exists t)[\{\alpha\}(x, t)\downarrow \;\&\; \{\beta\}(t)\downarrow]$$

is Σ^0_1 by the closure properties in (1) of Theorem 4.4, and so there is a recursive ε_0 such that

$$(\exists t)[\{\alpha\}(x, t)\downarrow \;\&\; \{\beta\}(t)\downarrow] \iff \{\varepsilon_0\}(\alpha, \beta, x)\downarrow \iff \{S(\varepsilon_0, \alpha, \beta)\}(x)\downarrow,$$

so it is enough to set $\boldsymbol{u}(\alpha, \beta) = S(\varepsilon_0, \alpha, \beta)$. ⊣

4D. The finite-order Borel and arithmetical pointclasses. There are two (coded) Borel pointclasses $\underset{\sim}{\mathbf{\Sigma}}^0_k, \underset{\sim}{\mathbf{\Pi}}^0_k$ for each $k \geq 1$ and they are defined by recursion, starting with the definitions in 4B for $k = 1$ and setting, succinctly,

$$\underset{\sim}{\mathbf{\Sigma}}^0_{k+1} = \exists^{\mathbb{N}} \underset{\sim}{\mathbf{\Pi}}^0_k, \quad \underset{\sim}{\mathbf{\Pi}}^0_{k+1} = \forall^{\mathbb{N}} \underset{\sim}{\mathbf{\Sigma}}^0_k. \tag{4-12}$$

In full detail: a pointset $S \subseteq X$ is $\underset{\sim}{\mathbf{\Sigma}}^0_{k+1}$ with code α, if there is a $\underset{\sim}{\mathbf{\Pi}}^0_k$ set $P \subseteq X \times \mathbb{N}$ with code α such that

$$x \in S \iff (\exists t)P(x, t); \tag{4-13}$$

and a pointset $P \subseteq X$ is $\underset{\sim}{\mathbf{\Pi}}^0_{k+1}$ with code α, if there is a $\underset{\sim}{\mathbf{\Sigma}}^0_k$ set $S \subseteq X \times \mathbb{N}$ with code α such that

$$x \in P \iff (\forall t)S(x, t). \tag{4-14}$$

$$\begin{array}{ccccccccccccccc}
 & & \underset{\sim}{\boldsymbol{\Sigma}}^0_1 & & & & \underset{\sim}{\boldsymbol{\Sigma}}^0_2 \cdots & & & & \underset{\sim}{\boldsymbol{\Sigma}}^1_1 & & & & \underset{\sim}{\boldsymbol{\Sigma}}^1_2 \cdots \\
 & \subsetneq & & \supsetneq & & \subsetneq & & & & \subsetneq & & \supsetneq & & \subsetneq & \\
\underset{\sim}{\boldsymbol{\Delta}}^0_1 & & & & \underset{\sim}{\boldsymbol{\Delta}}^0_2 & & & & \underset{\sim}{\boldsymbol{\Delta}}^1_1 & & & & \underset{\sim}{\boldsymbol{\Delta}}^1_2 & & \\
 & \subsetneq & & \supsetneq & & \subsetneq & & & & \subsetneq & & \supsetneq & & \subsetneq & \\
 & & \underset{\sim}{\boldsymbol{\Pi}}^0_1 & & & & \underset{\sim}{\boldsymbol{\Pi}}^0_2 \cdots & & & & \underset{\sim}{\boldsymbol{\Pi}}^1_1 & & & & \underset{\sim}{\boldsymbol{\Pi}}^1_2 \cdots
\end{array}$$

DIAGRAM 1. The finite-order Borel and the projective pointclasses.

The ***arithmetical pointclasses*** Σ^0_k and Π^0_k are the effective parts of these.[22]

LEMMA 4.7. *The finite-order Borel pointclasses are uniformly closed under continuous substitutions and satisfy the inclusions in the left-half of Diagram* 1.

PROOF. The first claim is proved by a trivial induction on k: e.g., if (4-13) holds and $f : Z \to X$ is continuous, then

$$S(f(z)) \iff (\exists t)P(f(z), t)$$

and the set $P^* = \{(z, t) \mid P(f(z), t)\}$ is $\underset{\sim}{\boldsymbol{\Pi}}^0_k$ by the induction hypothesis because $(z, t) \mapsto (f(z), t)$ is continuous. The second follows by using closure under recursive substitutions and trivial (vacuous) quantifications. ⊣

THEOREM 4.8. *For every* $k \geq 1$:

(1) $\underset{\sim}{\boldsymbol{\Sigma}}^0_k$ *and* Σ^0_k *have the same closure properties as* $\underset{\sim}{\boldsymbol{\Sigma}}^0_1$ *and* Σ^0_1, *and* $\underset{\sim}{\boldsymbol{\Pi}}^0_k$ *and* Π^0_k *have the same closure properties as* $\underset{\sim}{\boldsymbol{\Pi}}^0_1$ *and* Π^0_1 (as these are listed in (1) of Theorem 4.4).

(2) (LEM). *Both* $\underset{\sim}{\boldsymbol{\Sigma}}^0_k$ *and* $\underset{\sim}{\boldsymbol{\Pi}}^0_k$ *are closed under disjunction* $\vee$, *conjunction* & *and bounded quantification of both kinds,* $\exists^{\leq}, \forall^{\leq}$.

(3) *For every* X, $\underset{\sim}{\boldsymbol{\Sigma}}^0_k$ *and* $\underset{\sim}{\boldsymbol{\Pi}}^0_k$ *have universal sets* $G^{X,0}_k$ *and* $F^{X,0}_k$ *at* X.

(4) *Every* $\underset{\sim}{\boldsymbol{\Sigma}}^0_k$ *is closed under countable unions and every* $\underset{\sim}{\boldsymbol{\Pi}}^0_k$ *is closed under countable intersections.*

(5) *All the closure properties of these pointclasses in* (1), (2) *and* (4) *hold uniformly*; *for* (4), *for example, there is a recursive* $\boldsymbol{u} : \mathcal{N} \to \mathcal{N}$ *such that*

$$\bigcup_i \{x \mid G^{X,0}_k((\alpha)_i, x)\} = \{x \mid G^{X,0}_k(\boldsymbol{u}(\alpha), x)\}.$$

(6) (MP) *For every* X,

$$\neg\neg F^{X,0}_k(\alpha, x) \iff \neg G^{X,0}_k(\alpha, x). \tag{4-15}$$

(7) (LEM) *For every* X,

$$\neg\neg F^{X,0}_k(\alpha, x) \iff \neg G^{X,0}_k(\alpha, x) \iff F^{X,0}_k(\alpha, x). \tag{4-16}$$

PROOF. (1) is proved by induction on k, using the same equivalences we needed for $k = 1$ in the proof of (1) of Theorem 4.4.

(2) is (classically) routine, but cannot be proved from our assumptions.

[22]In the classical theory we also define the *self-dual* pointclasses $\underset{\sim}{\boldsymbol{\Delta}}^0_k = \underset{\sim}{\boldsymbol{\Sigma}}^0_k \cap \underset{\sim}{\boldsymbol{\Pi}}^0_k$ and we have included them in Diagram 1; however, very little can be proved about them in the intuitionistic system and so we will not discuss them further here.

(3) is proved by induction on k, starting with the definitions in Theorem 4.5 and setting in the induction step

$$G_{k+1}^{X,0}(\alpha, x) \iff (\exists t) F_k^{X\times\mathbb{N},0}(\alpha, x, t),$$
$$F_{k+1}^{X,0}(\alpha, x) \iff (\forall t) G_k^{X\times\mathbb{N},0}(\alpha, x, t).$$

(4) Suppose that for each i, $A_i \subseteq X$ and $A_i \in \utilde{\mathbf{\Sigma}}_k^0$ and let $G = G_k^{X,0}$ be universal for $\utilde{\mathbf{\Sigma}}_k^0$ at X. The Countable Axiom of Choice (AC_1^0) guarantees an α such that for each i, $A_i = \{x \mid G((\alpha)_i, x)\}$ so that $x \in \bigcup_i A_i \iff (\exists i) G((\alpha)_i, x)$ and $\bigcup_i A_i \in \utilde{\mathbf{\Sigma}}_k^0$ by closure under continuous substitutions.

(5) is proved using the universal sets as we did for $k = 1$. For the specific example of countable unions, check (skipping the superscripts) that the pointset

$$Q(\alpha, x) \iff (\exists i) G_k((\alpha)_i, x)$$

is in Σ_k^0 by the closure properties, and so

$$Q(\alpha, x) \iff (\exists i) G((\alpha)_i, x) \iff G(S^Q(\alpha), x)$$

with a recursive Q, so we can set $\boldsymbol{u}(\alpha) = S^Q(\alpha)$.

(6) is proved by induction on k, using the following fact essentially due to Solovay, cf. J. R. Moschovakis [2003]: *If $\phi(\alpha, x, t)$ is a formula in the language of* B *with no quantifiers over $\mathcal{N}$, then*

$$\mathsf{B} + \mathsf{MP} \vdash (\forall t)\neg\neg\phi(\alpha, x, t) \leftrightarrow \neg\neg(\forall t)\phi(\alpha, x, t).$$

(7) follows from (6) using LEM. ⊣

Corollary 4.9 (The finite-order hierarchy, MP). *The inclusions in the left-hand-side of Diagram* 1 *are all proper for $X = \mathcal{N}$; and the corresponding inclusions for the effective pointclasses are all proper for $X = \mathbb{N}$.*

Proof. For any $k \geq 1$, set

$$H^k(\alpha) \iff G_k^{\mathcal{N},0}(\alpha, \alpha) \text{ and } J^k(\alpha) \iff F_k^{\mathcal{N},0}(\alpha, \alpha).$$

H^k is in $\utilde{\mathbf{\Sigma}}_k^0$ by the closure properties; if it were also in $\utilde{\mathbf{\Pi}}_k^0$, then there would be some ε such that for all α,

$$H^k(\alpha) \iff F_k^{\mathcal{N},0}(\varepsilon, \alpha),$$

and so by (4-15), $\neg\neg H^k(\varepsilon) \iff \neg H^k(\varepsilon)$, which is impossible. Similarly, J^k is $\utilde{\mathbf{\Pi}}_k^0$ but not $\utilde{\mathbf{\Sigma}}_k^0$.

The argument for the second claim is similar. ⊣

Veldman [1990], [2008], [2009] develops an intricate theory of the Borel pointclasses of finite (and even infinite) order in a strong intuitionistic extension of B with (classically false) continuity principles, see Section 9B.

4E. Projective and analytical pointsets. The (coded) projective pointclasses are defined by recursion on $k \geq 0$, succinctly

$$\utilde{\mathbf{\Sigma}}_0^1 = \utilde{\mathbf{\Sigma}}_1^0, \quad \utilde{\mathbf{\Pi}}_0^1 = \utilde{\mathbf{\Pi}}_1^0, \quad \utilde{\mathbf{\Sigma}}_{k+1}^1 = \exists^{\mathcal{N}} \utilde{\mathbf{\Pi}}_k^1, \quad \utilde{\mathbf{\Pi}}_{k+1}^1 = \forall^{\mathcal{N}} \utilde{\mathbf{\Sigma}}_k^1;$$

a pointset is ***projective*** if it belongs to some $\utilde{\mathbf{\Sigma}}_k^1$ and ***analytical*** if it is Σ_k^1 for some k, i.e., if it is in some $\utilde{\mathbf{\Sigma}}_k^1$ with a recursive code.

It is easy to establish for these pointclasses the natural extensions of Lemma 4.7 and (1) – (5) and (7) of Theorem 4.8. Classically, the inclusions in the right-hand-side of Diagram 1 are proper and these pointsets admit a simple characterization:

LEMMA 4.10 (LEM). *For every $X = X_1 \times \cdots \times X_n$, a pointset $P \subseteq X$ is analytical if and only if it is definable by a formula in the language of* B, *and projective if it is definable by a formula with parameters from $\mathcal{N}$.*

Beyond that, very little can be proved about them for $k > 2$, even in full ZFC, because of fundamental consistency and independence results of Gödel and Cohen.[23]

Intuitionistically, the situation is much "worse": in a strong system (with classically false continuity axioms), Veldman [1990] shows that the projective hierarchy in Diagram 1 collapses to $\utilde{\mathbf{\Sigma}}^1_2$, cf. the discussion in Section 9B below.

Finally, about the classical theory, we should note that starting with Lusin [1925a], [1925b], [1925c] and Sierpinski [1925] which introduced them, full classical logic was used freely in the development of the theory of projective sets. In other words, the common assumption that the founders of descriptive set theory had some kind of coherent, constructive "universe" or, at least "approach" in mind (e.g., in Y. N. Moschovakis [2010a, Section 5]) cannot really be sustained: they worked in $\mathsf{B}^* + \mathsf{LEM}$, i.e., classical analysis, and for some of the results they claimed (like the hierarchy theorem for the projective pointclasses) they needed the full strength of classical analysis.

§5. Analytic and co-analytic sets.

Spelling out the definitions above for the most important, first two projective pointclasses using (4-4) and (4-3): a set $A \subseteq X$ is ***analytic*** (or $\utilde{\mathbf{\Sigma}}^1_1$) with code α if[24]

$$x \in A \iff (\exists \beta)[\{\alpha\}(x, \beta)\uparrow] \tag{5-1}$$

and $B \subseteq X$ is ***co-analytic*** (or $\utilde{\mathbf{\Pi}}^1_1$) with code α if

$$x \in B \iff (\forall \beta)[\{\alpha\}(x, \beta)\downarrow]. \tag{5-2}$$

Classically—in fact just assuming MP—the co-analytic ($\utilde{\mathbf{\Pi}}^1_1$) sets are exactly the complements of analytic ($\utilde{\mathbf{\Sigma}}^1_1$) sets, hence the terminology.

THEOREM 5.1. (1) *The pointclasses $\utilde{\mathbf{\Sigma}}^1_1$ and $\utilde{\mathbf{\Pi}}^1_1$ are uniformly closed under continuous substitutions and have universal sets at every X which induce their standard codings.*

(2) *$\utilde{\mathbf{\Sigma}}^1_1$ is uniformly closed under conjunction &, branching, disjunction $\vee$, both kinds of number quantification $\exists^{\mathbb{N}}$ and $\forall^{\mathbb{N}}$, and existential quantification $\exists^{\mathcal{N}}$ over $\mathcal{N}$.*

(3) *$\utilde{\mathbf{\Pi}}^1_1$ is uniformly closed under conjunction &, branching, universal quantification $\forall^{\mathbb{N}}$ over $\mathbb{N}$, and universal quantification $\forall^{\mathcal{N}}$ over $\mathcal{N}$.*

(4) *For every $k \geq 1$, $\utilde{\mathbf{\Sigma}}^0_k \cup \utilde{\mathbf{\Pi}}^0_k \subseteq \utilde{\mathbf{\Sigma}}^1_1$.*

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[23] The projective hierarchy has been studied extensively since the introduction of so-called *strong hypotheses* in the late 1960s, cf. Y. N. Moschovakis [2009] (and further references there).

[24] A is Σ^1_1 (***effectively analytic***) if it has a recursive $\utilde{\mathbf{\Sigma}}^1_1$-code, and similarly for the ***effectively co-analytic*** or Π^1_1 sets. If you are not already familiar with it, notice the unfortunate clash of terminologies:

$$\text{analytic} = \utilde{\mathbf{\Sigma}}^1_1, \quad \text{effectively analytic} = \Sigma^1_1, \quad \text{analytical} = \bigcup_k \Sigma^1_k.$$

There is a long, boring history which accounts for it, and people have learned to live with it.

(5) $\underset{\sim}{\mathbf{\Sigma}}^1_1$ *is uniformly closed under countable unions and intersections, and* $\underset{\sim}{\mathbf{\Pi}}^1_1$ *is uniformly closed under countable intersections.*

(6) *Parts* (1) – (4) *also hold for the lightface parts* $\Sigma^1_1, \Pi^1_1, \Sigma^0_k, \Pi^0_k$ *with "recursive" replacing "continuous" in* (1).

Proof. (1), (2), (3) and (5) are verified using the corresponding properties of $\underset{\sim}{\mathbf{\Pi}}^0_1, \underset{\sim}{\mathbf{\Sigma}}^0_1$ and basic equivalences such as:

$$\begin{aligned}
(\exists\beta)P(\beta)\ \&\ (\exists\gamma)Q(\gamma) &\iff (\exists\delta)[P((\delta)_0)\ \&\ Q((\delta)_1)]\\
(\exists\beta)P(\beta) \vee (\exists\gamma)Q(\gamma) &\iff (\exists\delta)[\text{if}\ \delta(0)=0\ \text{then}\ P(\delta^*)\ \text{else}\ Q(\delta^*)]\\
(\exists t)(\exists\beta)P(\beta,t) &\iff (\exists\beta)P(\beta^*,\beta(0))\\
(\forall t)(\exists\gamma)P(\gamma,t) &\iff (\exists\gamma)(\forall t)P((\gamma)_t,t) \quad (\text{using } (\mathrm{AC}^0_1))\\
(\exists\gamma)(\exists\beta)P(\gamma,\beta) &\iff (\exists\delta)P((\delta)_0,(\delta)_1)
\end{aligned}$$

These closure properties hold uniformly because we have universal sets, as above.

(4) We prove $\underset{\sim}{\mathbf{\Sigma}}^0_k \cup \underset{\sim}{\mathbf{\Pi}}^0_k \subseteq \underset{\sim}{\mathbf{\Sigma}}^1_1$ by induction on $k \geq 1$, using Lemma 4.3 at the basis with the trivial

$$\begin{aligned}
(\exists t)[f(\alpha,x,t)=0] &\iff (\exists\beta)(\forall s)[f(\alpha,x,\beta(0))=0]\\
(\forall t)[f(\alpha,x,t)\neq 0] &\iff (\exists\beta)(\forall t)[f(\alpha,x,t)\neq 0]
\end{aligned}$$

and the closure properties (1) and (2).

Finally, (6) holds because the closure properties hold uniformly and recursive functions preserve recursiveness by (3) of Theorem 3.1. ⊣

The most interesting properties of $\underset{\sim}{\mathbf{\Sigma}}^1_1$ and $\underset{\sim}{\mathbf{\Pi}}^1_1$ depend on the following representations of them using trees, as follows.

5A. Trees on $\mathbb{N}$. A ***tree on*** $\mathbb{N}$ with code $\tau \in \mathcal{N}$ is any set T of finite sequences from $\mathbb{N}$ satisfying the following:

(T1) The empty sequence $\emptyset$ is in T—its *root*.
(T2) T is closed under initial segments, i.e.,

$$[(u_0,\dots,u_{t-1}) \in T\ \&\ 0 < s < t] \Longrightarrow (u_0,\dots,u_{s-1}) \in T.$$

(T3) $u \in T \iff \tau(\langle u\rangle) = 0$.

The *body* of a tree T is the set of all infinite branches through it,

$$[T] = \{\alpha \mid (\forall t)[(\alpha(0),\dots,\alpha(t-1)) \in T]\} = \{\alpha \mid (\forall t)[\tau(\overline{\alpha}(t))=0]\}.$$

Theorem 5.2 (Normal Forms for $\underset{\sim}{\mathbf{\Sigma}}^1_1$). *For a pointset $A \subseteq X$, the following are equivalent*:

(i) *A is analytic.*

(ii) *There is a continuous function $g : X \to \mathcal{N}$ such that*:

(A1) *For every $x \in X$, $g(x)$ is a code of a tree $T(x)$ on $\mathbb{N}$, and*
(A2) $x \in A \iff (\exists\alpha)[\alpha \in [T(x)]] \iff (\exists\alpha)(\forall t)[g(x)(\overline{\alpha}(t))=0]$.

(iii) *There is a continuous $h : \mathbb{N}^2 \to \mathbb{N}$ with code $\widetilde{h} \in \mathcal{N}$ such that*

$$x \in A \iff (\exists\alpha)(\forall t)[\{\widetilde{h}\}(\overline{x}(t),\overline{\alpha}(t))=0].$$

Moreover, these characterizations are uniformly equivalent, i.e., $\widetilde{h}$ *and a code* $\widetilde{g}$ *of* g *can be recursively computed from any* $\underset{\sim}{\mathbf{\Sigma}}^1_1$ *code of* A, *and a* $\underset{\sim}{\mathbf{\Sigma}}^1_1$*-code of* A *can be recursively computed from* $\widetilde{h}$ *and from any code of* g.

The last claim implies that if A is effectively analytic (Σ^1_1), then g and $\widetilde{h}$ can be chosen to be recursive.

Proof. (i) $\Rightarrow$ (ii). If A is analytic with code β, then by (5-1) and (3) of Theorem 3.2,

$$\begin{aligned} x \in A \iff (\exists\alpha)[\{\beta\}(x,\alpha)\uparrow] &\iff (\exists\alpha)[\{S(\beta,x)\}(\alpha)\uparrow] \\ &\iff (\exists\alpha)(\forall t)[S(\beta,x)(\overline{\alpha}(t)) = 0]. \end{aligned}$$

We get the required $g : X \to \mathcal{N}$ (and a code $\widetilde{g}$ of it) easily from $S(\beta, x)$.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). If $g : X \to \mathcal{N}$ with code $\widetilde{g}$ satisfies (ii), then the required $\widetilde{h}$ can be computed from $\widetilde{g}$, and a $\underset{\sim}{\mathbf{\Sigma}}^1_1$-code of A can be computed from any $\widetilde{h}$ satisfying (iii). ⊣

For the corresponding result for $\underset{\sim}{\mathbf{\Pi}}^1_1$ we need the notion of a grounded tree: a tree T on $\mathbb{N}$ with code τ is ***grounded*** if its body is *positively empty*, i.e.,

(5-3) $\qquad T$ with code τ is grounded $\iff (\forall\alpha)(\exists t)[\tau(\overline{\alpha}(t)) \neq 0]$.

Theorem 5.3 (Normal Forms for $\underset{\sim}{\mathbf{\Pi}}^1_1$). *For a pointset* $B \subseteq X$, *the following are equivalent*:

(i) *B is co-analytic.*

(ii) *There is a continuous* $g : X \to \mathcal{N}$ *such that*

(CA1) *For every* $x \in X$, $g(x)$ *is a code of a tree* $T(x)$ *on* $\mathbb{N}$, *and*
(CA2) $x \in B \iff T(x)$ *is grounded* $\iff (\forall\alpha)(\exists t)[g(x)(\overline{\alpha}(t)) \neq 0]$.

(iii) *There is a continuous* $h : \mathbb{N}^2 \to \mathbb{N}$ *with code* $\widetilde{h} \in \mathcal{N}$ *such that*

$$x \in B \iff (\forall\alpha)(\exists t)[\{\widetilde{h}\}(\overline{x}(t), \overline{\alpha}(t)) \neq 0].$$

Moreover, these equivalences hold uniformly (as in Theorem 5.2).

Proof is just like that of the preceding theorem. ⊣

5B. Bar induction and bar recursion. The following basic fact is (**B**3) of our assumptions in Section 2. It is justified by Kleene and Vesley [1965, *26.8a] (with $u \notin T$ as their $R(u)$):

Theorem 5.4 (Proof by bar induction). *Suppose T is a grounded tree on $\mathbb{N}$ and P is a relation on finite sequences from $\mathbb{N}$ such that*

(1) *if* $(u_0, \ldots, u_{t-1}) \notin T$, *then* $P(u_0, \ldots, u_{t-1})$, *and*
(2) *for every* $(u_0, \ldots, u_{t-1}) \in T$,

$$(\forall v) P(u_0, \ldots, u_{t-1}, v) \Longrightarrow P(u_0, \ldots, u_{t-1});$$

it follows that $P(u_0, \ldots, u_{t-1})$ *holds for every sequence and in particular* $P(\emptyset)$.

Using bar induction, it is possible to justify very strong *definitions by bar recursion* on a grounded tree; we will only need the following "continuous" result of this type, which is easy to prove by appealing to the 2nd Recursion Theorem:

THEOREM 5.5 (Effective bar recursion). *Suppose T is a grounded tree and*

$$h_0 : \mathbb{N} \times X \to \mathcal{N}, \quad h_1 : \mathcal{N} \times X \to \mathcal{N}$$

are continuous; *then there exists a continuous $h : \mathbb{N} \times X \to \mathcal{N}$ such that for all sequences u* (with the notation in Footnote 11)

$$h(\langle u \rangle, x) = \begin{cases} h_0(\langle u \rangle, x) & \text{if } u \notin T, \\ h_1(\langle\!\langle h(\langle u \rangle * \langle i \rangle, x) \mid i < \infty \rangle\!\rangle, x) & \text{otherwise.} \end{cases}$$

Moreover, this holds uniformly, i.e., a code for h can be computed from any codes of h_0, h_1 and T, so if they are recursive, then so is h.

PROOF. We set

$$f(\varepsilon, t, x) = \begin{cases} h_0(t, x), & \text{if } \neg[t = \langle u \rangle \text{ for some } u \in T], \\ h_1(\langle\!\langle \{\varepsilon\}^{\mathbb{N} \times X, 1}(t * \langle i \rangle, x) \mid i < \infty \rangle\!\rangle, x) & \text{otherwise.} \end{cases}$$

This is a continuous partial function, so by the 2nd Recursion Theorem, there is an $\widetilde{\varepsilon}$ such that

$$f(\widetilde{\varepsilon}, t, x)\downarrow \implies f(\widetilde{\varepsilon}, t, x) = \{\widetilde{\varepsilon}\}(t, x);$$

and if we set $h(t, x) = \{\widetilde{\varepsilon}\}(t, x)$, then the definition of f yields

$$h(t, x)\downarrow$$

$$\implies h(t, x) = \begin{cases} h_0(t, x), & \text{if } \neg[t = \langle u \rangle \text{ for some } u \in T], \\ h_1(\langle\!\langle h(t * \langle i \rangle, x) \mid i < \infty \rangle\!\rangle, x) & \text{otherwise.} \end{cases}$$

It suffices to prove that for each $x \in X$,

$$\text{for all sequences } u, h(\langle u \rangle, x)\downarrow,$$

and this is done by bar induction on T. ⊣

§6. Inductive definitions and proofs on $\mathcal{N}$. Definitions by induction and inductive proofs over such definitions were accepted by Brouwer and the early descriptive set theorists. We prove here that a simple—but very useful—case of this method can be justified from our assumptions, and in 6A we discuss briefly some relevant classical results.[25]

THEOREM 6.1 (Π^0_1-inductive definitions). *For any two continuous functions $g_0 : \mathcal{N} \to \mathbb{N}$ and $g_1 : \mathcal{N} \times \mathbb{N} \to \mathcal{N}$, there is a unique set $I \subseteq \mathcal{N}$ with the following properties*:

(I1) $(\forall \alpha)\Big(\Big[g_0(\alpha) = 0 \vee [g_0(\alpha) \neq 0 \,\&\, (\forall i)[g_1(\alpha, i) \in I]]\Big] \implies \alpha \in I\Big)$.

(I2) *If $P \subseteq \mathcal{N}$ satisfies* (I1) *with $I := P$, i.e.,*

$$(\forall \alpha)\Big(\Big[g_0(\alpha) = 0 \vee [g_0(\alpha) \neq 0 \,\&\, (\forall i)[g_1(\alpha, i) \in P]]\Big] \implies \alpha \in P\Big),$$

then $I \subseteq P$.

(I3) *I is $\underset{\sim}{\Pi}{}^1_1$.*

[25]Veldman [2008, Section 1.5] postulates a version of Theorem 6.1 without (I3).

Moreover, a $\underset{\sim}{\mathbf{\Pi}}^1_1$*-code for* I *can be recursively computed from codes of* g_0 *and* g_1*, and so if these are recursive, then* I *is* Π^1_1.

It is easy to check (classically) that there is a *least set* I which satisfies the *fixed-point equivalence*

$$\alpha \in I \iff \Big(g_0(\alpha) = 0 \vee [g_0(\alpha) \neq 0 \;\&\; (\forall i)[g_1(\alpha, i) \in I]]\Big),$$

which is, of course, the set we need, but it is not quite immediate—even classically—that I is $\underset{\sim}{\mathbf{\Pi}}^1_1$ or that we can justify (I2) in $\mathbf{B}$. The key observation is that

$$\alpha \in I \iff \mathcal{S}_\alpha \text{ is grounded},$$

where $\alpha \mapsto \mathcal{S}_\alpha$ is a (suitably) continuous map which assigns to each α a *tree on* $\mathcal{N}$ *all of whose nodes can be computed recursively from* α; and the gist of the proof of the theorem from our assumptions is to use this fact to replace $\mathcal{S}_\alpha$ by a tree S_α on $\mathbb{N}$, so that we can then use bar recursion.

We give the proof in five Lemmas, without worrying about the “moreover” claim which follows from the argument.

First, define for each $n \geq 0$ a continuous function $f_n : \mathbb{N}^n \times \mathcal{N} \to \mathcal{N}$ by the following recursion:

$$f_0(\alpha) = \alpha, \quad f_{n+1}(t_1, \dots, t_{n+1}, \alpha) = g_1(f_n(t_1, \dots, t_n, \alpha), t_{n+1}) \tag{6-1}$$

so that, for example

$$f_1(t_1, \alpha) = g_1(\alpha, t_1) \quad f_2(t_1, t_2, \alpha) = g_1(f_1(t_1, \alpha), t_2) = g_1(g_1(\alpha, t_1), t_2), \dots$$

It simplifies the notation to think of this definition as providing a sequence $f_0, f_1, \dots$ of functions with different arguments, but in fact

$$f_n(t_1, \dots, t_n, \alpha) = f(\langle t_1, \dots, t_n \rangle, \alpha) \tag{6-2}$$

with a single, continuous $f : \mathbb{N} \times \mathcal{N} \to \mathcal{N}$, which is what we need to use these functions in further computations. We skip the simple proof.

For the main construction, we set for each α

$$(t_1, \dots, t_n) \in S_\alpha \iff n = 0 \vee (\forall i < n)[g_0(f_i(t_1, \dots, t_i, \alpha)) \neq 0]. \tag{6-3}$$

Lemma 1. For every α*,* S_α *is a tree on* $\mathbb{N}$ *with code which can be computed continuously from* α.

This is where the “uniform” definition of the f_n’s in (6-2) is needed.

We set

$$\alpha \in I \iff S_\alpha \text{ is grounded}, \tag{6-4}$$

and the Normal Forms Theorem 5.3 gives immediately

Lemma 2 (I3). I *is* $\underset{\sim}{\mathbf{\Pi}}^1_1$.

To prove (I1) and (I2) we will need the following identities:

Lemma 3. For all $n \geq 0$,

$$f_{n+1}(t_1, \dots, t_{n+1}, \alpha) = f_n(t_2, \dots, t_{n+1}, g_1(\alpha, t_1)). \tag{6-5}$$

PROOF is by induction on n, trivial at the basis because

$$f_1(t_1, \alpha) = g_1(\alpha, t_1) = f_0(g_1(\alpha, t_1)).$$

In the inductive step,

$$\begin{aligned} f_{n+2}(t_1, \dots, t_{n+2}, \alpha) &= g_1(f_{n+1}(t_1, \dots, t_{n+1}, \alpha), t_{n+2}) \quad \text{(by def of } f_{n+2}) \\ &= g_1\Big(f_n(t_2, \dots, t_{n+1}, g_1(\alpha, t_1)), t_{n+2}\Big) \quad \text{(ind hyp)} \\ &= f_{n+1}(t_2, \dots, t_{n+2}, g_1(\alpha, t_1)) \quad \text{(by def of } f_{n+1}). \quad \dashv \end{aligned}$$

Lemma 4 (I1). *For any α, if either $g_0(\alpha) = 0$ or $g_0(\alpha) \neq 0$ & $(\forall i)[g_1(\alpha, i) \in I]$, then $\alpha \in I$.*

PROOF. If $g_0(\alpha) = 0$, then the only tuple in S_α is the root, i.e., $S_\alpha = \{\emptyset\}$ and so S_α is grounded.

Suppose now that $g_0(\alpha) \neq 0$ and for all i, $S_{g_1(\alpha,i)}$ is grounded. To prove that S_α is grounded, we must show that

$$(\forall \gamma)(\exists t)[(\gamma(0), \gamma(1), \dots, \gamma(t)) \notin S_\alpha].$$

To see this, fix γ, let $j = \gamma(0)$ and notice that $(j) \in S_\alpha$, since $g_0(\alpha) \neq 0$. By the definitions and Lemma 3 then, we have that for every $n \geq 2$,

$$\begin{aligned} (t_2, \dots, t_n) \in S_{g_1(\alpha,j)} &\iff (\forall i < n-1)[g_0(f_i(t_2, \dots, t_{i+1}, g_1(\alpha, j))) \neq 0] \\ &\iff (\forall i < n-1)[g_0(f_{i+1}(j, t_2, \dots, t_{i+1}, \alpha)) \neq 0] \\ &\iff (\forall i < n)[g_0(f_i(j, t_2, \dots, t_i, \alpha)) \neq 0] \\ &\iff (j, t_2, \dots, t_n) \in S_\alpha. \end{aligned}$$

Since $S_{g_1(\alpha,j)}$ is grounded, there is some t such that

$$(\gamma(1), \gamma(2), \dots, \gamma(t)) \notin S_{g_1(\alpha,j)};$$

and then $(\gamma(0), \gamma(1), \gamma(2), \dots, \gamma(t)) \notin S_\alpha$, which completes the proof. $\dashv$

Lemma 5 (I2). *If $P \subseteq \mathcal{N}$ and*

$$(\forall \alpha)\Big([g_0(\alpha) = 0 \vee [g_0(\alpha) \neq 0 \;\&\; (\forall i)[g_1(\alpha, i) \in P]]] \Longrightarrow \alpha \in P\Big)$$

then $I \subseteq P$.

PROOF. We assume that $\alpha \in I$, so that S_α is grounded, set

$$\begin{aligned} P^*(t_1, \dots, t_n) \iff &(t_1, \dots, t_n) \notin S_\alpha \\ &\vee [(t_1, \dots, t_n) \in S_\alpha \;\&\; f_n(t_1, \dots, t_n, \alpha) \in P], \end{aligned}$$

and then, using the hypothesis on P and Lemma 4, we prove by bar induction that $(\forall u)P^*(u)$, which for $u = \emptyset$ yields the required $f_0(\alpha) = \alpha \in P$. $\dashv$

Using Theorem 6.1 and the 2nd Recursion Theorem, we can justify recursive definitions over inductively defined subsets of $\mathcal{N}$.

THEOREM 6.2 (Effective grounded recursion). *Suppose* $g_0, h_0, h_1 : \mathcal{N} \to \mathcal{N}$ *and* $g_1 : \mathcal{N} \times \mathbb{N} \to \mathcal{N}$ *are recursive functions and* $I \subseteq \mathcal{N}$ *is the set determined inductively from* g_0 *and* g_1. *There is a recursive partial function* $h : \mathcal{N} \rightharpoonup \mathcal{N}$, *such that* (with the notation in Footnote 11)

$$\alpha \in I \Longrightarrow h(\alpha) = \begin{cases} h_0(\alpha), & \textit{if } g_0(\alpha) = 0, \\ h_1(\langle\!\langle h(g_1(\alpha, i)) \mid i < \infty \rangle\!\rangle), & \textit{otherwise}; \end{cases} \tag{6-6}$$

in particular, $\alpha \in I \Longrightarrow h(\alpha)\downarrow$.

PROOF. We set

$$f(\varepsilon, \alpha) = \begin{cases} h_0(\alpha), & \text{if } g_0(\alpha) = 0, \\ h_1(\langle\!\langle \{\varepsilon\}^{\mathcal{N},1}(g_1(\alpha, i)) \mid i < \infty \rangle\!\rangle), & \text{otherwise.} \end{cases} \tag{6-7}$$

This is a recursive partial function, so by the 2nd Recursion Theorem, there is a recursive $\widetilde{\varepsilon}$ such that

$$f(\widetilde{\varepsilon}, \alpha)\downarrow \Longrightarrow f(\widetilde{\varepsilon}, \alpha) = \{\widetilde{\varepsilon}\}(\alpha); \tag{6-8}$$

and if we set $h(\alpha) = \{\widetilde{\varepsilon}\}(\alpha)$, then (6-7) yields

$$f(\widetilde{\varepsilon}, \alpha) = \begin{cases} h_0(\alpha), & \text{if } g_0(\alpha) = 0, \\ h_1(\langle\!\langle h(g_1(\alpha, i)) \mid i < \infty \rangle\!\rangle), & \text{otherwise,} \end{cases} \tag{6-9}$$

and the fixed-point condition (6-8) becomes

$$g_0(\alpha) = 0 \Longrightarrow h(\alpha) = h_0(\alpha),$$
$$g_0(\alpha) \neq 0 \;\&\; (\forall i)[h(g_1(\alpha, i))\downarrow] \Longrightarrow h(\alpha) = h_1(\langle\!\langle h(g_1(\alpha, i)) \mid i < \infty \rangle\!\rangle).$$

These last, two implications yield the required (6-6) by induction on the definition of I, (I2) of Theorem 6.1. ⊣

6A. Monotone induction and ordinal recursion. Theorem 6.1 is a (very) special case of the *Normed Induction Theorem*,[26] which derives explicit forms for the least-fixed-points of very general inductive operations. We will not state it here because it is rather complex and it cannot be proved from our assumptions, but it is rich in consequences in the classical theory, among them a simple proof of the following version of the *Cantor-Bendixson Theorem*, an early jewel of effective descriptive set theory:

THEOREM 6.3 (LEM, Kreisel [1959]). *If* $F \subseteq X$ *is closed* ($\underset{\sim}{\mathbf{\Pi}}^0_1$) *and* Σ^1_1 *and if*

$$F = k(F) \cup s(F)$$

is the canonical (unique) *decomposition of* F *into a perfect kernel* $k(F)$ *and a countable scattered part* $s(F)$, *then* $k(F)$ *is* Σ^1_1.

Kreisel also showed that *there is a* Π^0_1 *set* F *whose perfect kernel is not in* Π^1_1, so that from the definability point of view, this version of the *Cantor-Bendixson Theorem* is optimal.[27]

[26]Cf. Y. N. Moschovakis [2009, 7C.8] and Y. N. Moschovakis [2010b, Section 10.2], which includes a full proof and the application to Kreisel's Theorem.

[27]The Cantor-Bendixson Theorem has been considered by intuitionists, including Brouwer (very early) and Veldman [2009] more recently. We do not know an intuitionistic proof of a "natural" classical version of it, and we have nothing useful to say about this interesting problem.

The only facts about $\underset{\sim}{\mathbf{\Pi}}^1_1$ that come up in the proof of the Normed Induction Theorem are that it is uniformly closed under continuous substitutions, that it has universal sets and that it is *normed*, or has the *Prewellordering Property* in the standard (if awful) terminology.[28] We will also not try to explain this here as it, too, is complex and cannot be proved from our assumptions. One classical proof of it uses Theorem 5.3 to associate with each $\underset{\sim}{\mathbf{\Pi}}^1_1$ pointset A the "norm"

$$\pi(x) = \text{the ordinal rank of } T(x) \quad (\pi : A \to \text{Ords}),$$

which is constructive enough despite the reference to ordinals, cf. Veldman [2008]; but to prove then that it is a "$\underset{\sim}{\mathbf{\Pi}}^1_1$-norm", you need to *compare the ordinal ranks of grounded trees*, and there is the non-constructive rub. It is difficult to see how one can make real progress in the intuitionistic study of analytic and co-analytic sets without some version or substitute for the Prewellordering Property, perhaps about $\underset{\sim}{\mathbf{\Sigma}}^1_1$ which is constructively better behaved. We have no clue how to approach this interesting problem.

§7. **The coded pointclass of Borel sets.** As an immediate consequence of Theorem 6.1, we get:

THEOREM 7.1 (Borel codes). *There is a set* $\mathrm{BC} \subset \mathcal{N}$ *with the following properties*:

(BC1) $(\forall\alpha)\Big([\alpha(0) \leq 1 \vee (\forall i)[(\alpha^*)_i \in \mathrm{BC}]] \Longrightarrow \alpha \in \mathrm{BC}\Big)$.

(BC2) *If* $P \subseteq \mathcal{N}$ *and* $(\forall\alpha)\Big([\alpha(0) \leq 1 \vee (\forall i)[(\alpha^*)_i \in P]] \Longrightarrow \alpha \in P\Big)$, *then* $\mathrm{BC} \subseteq P$.

(BC3) BC *is* Π^1_1.

We now come to the definition of the Borel subsets of any given space X, which is to be given by induction on the set BC. We give first an informal definition of a map on BC to the "powerset" $\mathcal{P}(X)$ of a given space for which it is not immediately obvious how it can be made precise from our assumptions, since we do not have partial functions on $\mathcal{N}$ to $\mathcal{P}(X)$ in our setup; we will follow this by the official, rigorous definition.

Borel sets, informally. For a fixed space $X = X_1 \times \cdots \times X_n$ as usual and each $\alpha \in \mathrm{BC}$, we define the ***Borel subset of*** X ***with code*** α

$$B_\alpha = B^X_\alpha \subseteq X$$

by the following recursive clauses:

(BC0) If $\alpha(0) = 0$, $B_\alpha = \{x \mid \{\alpha^*\}(x)\downarrow\}$ = the $\underset{\sim}{\mathbf{\Sigma}}^0_1$ subset of X with code α^*.
(BC1) If $\alpha(0) = 1$, $B_\alpha = \{x \mid \{\alpha^*\}(x)\uparrow\}$ = the $\underset{\sim}{\mathbf{\Pi}}^0_1$ subset of X with code α^*.
(BC2) If $\alpha(0) = 2$ and $\alpha \in \mathrm{BC}$, then $B_\alpha = \bigcup_i B_{(\alpha^*)_i}$.
(BC3) If $\alpha(0) > 2$ and $\alpha \in \mathrm{BC}$, then $B_\alpha = \bigcap_i B_{(\alpha^*)_i}$.

A pointset $A \subseteq X$ is ***Borel*** (measurable) with code α if $A = B^X_\alpha$ for some $\alpha \in \mathrm{BC}$.

If we do not worry about justifying this from our assumptions, we can recognize it as a valid definition by recursion on the set BC of Borel codes; this is the clue to

[28] Cf. Y. N. Moschovakis [2009, Section 4B], [2010b, Section 10.2].

formulating it rigorously, as an application of Theorem 6.2 which assigns to each $\alpha \in \mathrm{BC}$ some $\boldsymbol{u}(\alpha) \in \mathcal{N}$ such that (BC0) – (BC3) hold if we set

$$B^X_\alpha = \text{ the } \underset{\sim}{\boldsymbol{\Sigma}}{}^1_1\text{-subset of } X \text{ with code } \boldsymbol{u}(\alpha).$$

The formulation and proof of the next result lean heavily on Theorem 5.1.

THEOREM 7.2 (Borel sets, rigorously). *Fix a space X and a set $G \subseteq \mathcal{N} \times X$ in Σ^1_1 which is universal for $\underset{\sim}{\boldsymbol{\Sigma}}{}^1_1$ at X* (by Theorem 5.1). *There is a recursive partial function $\boldsymbol{u}^X = \boldsymbol{u} : \mathcal{N} \rightharpoonup \mathcal{N}$ with the following properties*:

(1) *The domain of convergence of $\boldsymbol{u}$ includes* BC.

(2) (BC0) – (BC3) *above hold with $B^X_\alpha = B_\alpha = G_{\boldsymbol{u}(\alpha)}$.*

OUTLINE OF PROOF. We define $\boldsymbol{u}(\alpha)$ by grounded recursion, Theorem 6.2, after we check that the needed functions are recursive by appealing to the uniform closure properties of $\underset{\sim}{\boldsymbol{\Sigma}}{}^1_1$ listed in Theorem 5.1. ⊣

This result gives a rigorous definition of *the Borel subset B^X_α of X* for each $\alpha \in \mathrm{BC}$ and also proves that uniformly, *every Borel subset of X is $\underset{\sim}{\boldsymbol{\Sigma}}{}^1_1$*. It is then a simple matter to develop the theory of Borel sets as constructively as possible, starting with this:

THEOREM 7.3. (1) *Every $\underset{\sim}{\boldsymbol{\Sigma}}{}^0_k$ and every $\underset{\sim}{\boldsymbol{\Pi}}{}^0_k$ pointset is Borel, uniformly.*

(2) *The coded pointclass* B *of Borel sets is uniformly closed under continuous substitutions, conjunction* &, *disjunction* $\vee$, *both kinds of number quantification* $\exists^{\mathbb{N}}$ *and* $\forall^{\mathbb{N}}$ *and both countable unions and countable intersections.*

(3) (LEM) B *is uniformly closed under complementation.*

PROOF. (1) is immediate, by induction on k.

(2) To prove that B is closed under continuous substitution, we fix a continuous $f : X \to Y$ and we prove by induction on the definition of BC that

$$\alpha \in \mathrm{BC} \Longrightarrow (\exists \beta)[\beta \in \mathrm{BC} \;\&\; B^X_\beta = f^{-1}[B^Y_\alpha]].$$

This is known in cases (BC0) and (BC1). For (BC2), if $\alpha(0) = 2$, then the induction hypothesis (with (AC^0_1)) gives us a β such that for each i,

$$(\beta)_i \in \mathrm{BC} \;\&\; B^X_{(\beta)_i} = f^{-1}[B^Y_{(\alpha^*)_i}];$$

and then $\gamma = (2)\hat{\ }\beta \in \mathrm{BC}$ and

$$B^X_\gamma = \textstyle\bigcup_i B^X_i = \bigcup_i f^{-1}[B^Y_{(\alpha^*)_i}] = f^{-1}[B^Y_\alpha].$$

The argument for case (BC3) is similar and the other claims in (2) are simple.

(3) is proved using Theorem 6.2 (effective grounded recursion) on $\alpha \in \mathrm{BC}$ again and the De Morgan Laws, $X \setminus (\bigcap_i A_i) = \bigcup_i (X \setminus A_i)$ and its dual. ⊣

§8. The Separation and Suslin-Kleene Theorems. Two pointsets are disjoint if it is absurd that they have a common point,

$$A \cap B = \emptyset \iff \neg(\exists x)[x \in A \;\&\; x \in B] \quad (A, B \subseteq X).$$

THEOREM 8.1 (Strong Separation, MP). *For each space X, there is a recursive function $\boldsymbol{u} : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ such that if $A, B \subseteq X$ are disjoint $\underset{\sim}{\boldsymbol{\Sigma}}{}^1_1$ sets with codes α and β, then $\boldsymbol{u}(\alpha, \beta)$ is a Borel code of a set C which separates them, i.e.,*

$$A \subseteq C \text{ and } C \cap B = \emptyset.$$

The theorem is proved by writing out carefully the so-called "constructive proof" of the classical Separation Theorem of Lusin which ultimately defines the required $\boldsymbol{u}(\alpha, \beta)$ by Bar Recursion, Theorem 5.5. We will not reproduce any of the much-published versions of the argument.[29] It is worth, however, to set out the first part of the proof where MP is used.

We assume for simplicity that $X = \mathcal{N}$—and it is, in fact, quite simple to reduce the general case to this.

By the Normal Forms Theorem 5.2 for $\underset{\sim}{\boldsymbol{\Sigma}}^1_1$, we have representations

$$\begin{aligned} x \in A &\iff (\exists \gamma)(\forall t)[\{\widetilde{f}\}(\overline{x}(t), \overline{\gamma}(t)) = 0] \quad (x \in \mathcal{N}), \\ x \in B &\iff (\exists \delta)(\forall t)[\{\widetilde{h}\}(\overline{x}(t), \overline{\delta}(t)) = 0], \end{aligned}$$

with suitable $\widetilde{f}, \widetilde{h} \in \mathcal{N}$ which can be computed from α, β. The set

$$\begin{aligned} T = \Big\{ &(\langle x_0, c_0, d_0 \rangle, \dots, \langle x_{t-1}, c_{t-1}, d_{t-1} \rangle) \\ &\quad \mid (\forall i < t)\Big(\{\widetilde{f}\}(\langle x_0, \dots, x_i \rangle, \langle c_0, \dots, c_i \rangle) = 0 \\ &\qquad\qquad \& \ \{\widetilde{h}\}(\langle x_0, \dots, x_i \rangle, \langle d_0, \dots, d_i \rangle) = 0\Big)\Big\} \end{aligned}$$

is a tree on $\mathbb{N}$, and the first part of the argument is to show that it is grounded, which easily means that

$$(\forall x, \gamma, \delta)(\exists t)[(\langle x(0), \gamma(0), \delta(0) \rangle, \dots, \langle x(t-1), \gamma(t-1), \delta(t-1) \rangle) \notin T];$$

now Markov's Principle implies that this is equivalent to

$$(\forall x, \gamma, \delta)\neg(\forall t)[(\langle x(0), \gamma(0), \delta(0) \rangle, \dots, \langle x(t-1), \gamma(t-1), \delta(t-1) \rangle) \in T];$$

and this follows from the hypothesis $A \cap B = \emptyset$, because if for some x, γ, δ

$$(\forall t)[(\langle x(0), \gamma(0), \delta(0) \rangle, \dots, \langle x(t-1), \gamma(t-1), \delta(t-1) \rangle) \in T],$$

then $x \in A \cap B$.

Corollary 8.2 (The Suslin-Kleene Theorem, MP). *If both $A \subseteq X$ and its complement $A^c = X \setminus A$ are $\underset{\sim}{\boldsymbol{\Sigma}}^1_1$, then (uniformly) there is a Borel set B such that $A \subseteq B \subseteq A^{cc}$; and so if, also, $A^{cc} = A$, then A is Borel.*[30]

Proof. The theorem gives a Borel set B such that

$$A \subseteq B \text{ and } \neg(\exists x)[x \in B \ \& \ x \in A^c];$$

and the second of these implies $(\forall x)[x \in B \Longrightarrow \neg(x \in A^c)]$, i.e., $B \subseteq A^{cc}$. ⊣

This is a very weak constructive version of the classical Suslin Theorem since the "stability" assumption $A^{cc} = A$ does not come cheap, so it is worth noticing that it yields the first—and still one of the best—applications of the result.

A function $f : X \to W$ is ***Borel measurable*** with code α if its *nbhd diagram*

$$G_f(x, s) = \{(x, s) \mid f(x) \in N^W_s\} \subset X \times \mathbb{N}$$

[29]See, for example, Y. N. Moschovakis [2009, 7B.3] (and the comments in Y. N. Moschovakis [2010b, Section 10.1]) and also Veldman [2008] and Aczel [2009], who prove somewhat weaker versions of these results to avoid using MP.

[30]We are grateful to the referee for catching the carelessly written version of this result without the needed, additional hypothesis that $A^{cc} = A$.

is Borel with code α.

Corollary 8.3 (MP). *If* $f : X \to \mathcal{N}$ *and* Graph$(f) = \{(x, \beta) \mid f(x) = \beta\}$ *is* $\underset{\sim}{\mathbf{\Sigma}}^1_1$, *then* (*uniformly*) f *is Borel measurable.*

Lebesgue [1905] (essentially) claims this result for the special case where

$$f(x) = \text{the unique } \beta \text{ such that } g(x, \beta) = 0$$

with a Borel measurable $g : X \times \mathcal{N} \to \mathbb{N}$ such that $(\forall x)(\exists! \beta)[g(x, \beta) = 0]$ and gives an (in)famous wrong proof of it; the discovery of the error by (a very young) Suslin some ten years later and how it led him to the formulation and proof of the Suslin Theorem is an oft-told legend in the history of descriptive set theory, cf. Y. N. Moschovakis [2009] and Lebesgue's introduction to Lusin [1930].

Proof. The nbhd diagram G_f of f and its complement are easily analytic, using the closure properties of $\underset{\sim}{\mathbf{\Sigma}}^1_1$ and the trivial equivalences

$$\begin{aligned} f(x) \in N_s^{\mathcal{N}} &\iff (\exists \beta)[f(x) = \beta \;\&\; \beta \in N_s], \\ f(x) \notin N_s^{\mathcal{N}} &\iff (\exists \beta)[f(x) = \beta \;\&\; \beta \notin N_s]. \end{aligned}$$

To prove that $G_f^{cc} = G_f$ we verify that

$$\begin{aligned} f(x) \in N_s^{\mathcal{N}} &\iff (\forall \beta)[f(x) = \beta \Longrightarrow \beta \in N_s] \\ &\qquad\iff \neg(\exists \beta)[f(x) = \beta \;\&\; \beta \notin N_s], \end{aligned}$$

the first of these trivially and the second using the fact that $\{(\beta, s) \mid \beta \in N_s\}$ is recursive and hence $\neg\neg(\beta \in N_s) \iff \beta \in N_s$; so $G_f = B^c$ for some B, and hence $G_f^{cc} = B^{ccc} = B^c = G_f$ as required by the third hypothesis of the Suslin-Kleene Theorem, which then implies that G_f is Borel. $\dashv$

§9. Concluding remarks. We finish with a discussion of some important metamathematical properties of intuitionistic systems and their relevance for (effective) descriptive set theory.

9A. Realizability. Kleene [1945] and Nelson [1947] introduced *realizability interpretations* for intuitionistic arithmetic and then Kleene extended them to intuitionistic analysis in Kleene and Vesley [1965]. The idea is to interpret compound sentences as incomplete communications of "effective procedures" by which their correctness might be established. For arithmetic we can model these procedures by recursive partial functions on $\mathbb{N}$, coded by their Gödel numbers; for analysis, Kleene allowed continuous partial functions, coded in $\mathcal{N}$ much as we have coded them in Section 3. He proved that *every theorem of the full, non-classical intuitionistic analysis* $\mathbf{I}$ *is realized by a total recursive function*, which (among other things) establishes the consistency of $\mathbf{I}$, since $0 = 1$ is not realizable. In Kleene [1969] he obtained stronger results by formalizing realizability and its variant q-realizability in a conservative extension of $\mathbf{B}$.

The next result follows easily from (the proofs of) Theorems 50 and 53 and Remark 54 in Kleene [1969]. We use the notation of (2-1).

THEOREM 9.1. (1) *If* $\phi(x_1, \dots, x_n, \beta)$ *is a formula in the language of* B (whose free variables are all in the list $x = (x_1, \dots, x_m), \beta)$ *and*

$$\mathsf{B} \vdash (\forall x)(\exists \beta)\phi(x, \beta), \tag{9-1}$$

then

$$\mathsf{B} \vdash (\exists \varepsilon)[\mathrm{GR}(\varepsilon) \,\&\, (\forall x)(\exists \beta)[\{\varepsilon\}^{X,1}(x) = \beta \,\&\, \phi(x, \beta)]]. \tag{9-2}$$

Since B is classically sound, this implies:

(2) *If* $R \subseteq X \times \mathcal{N}$ *is the relation defined by* $\phi(x, \beta)$ *as above and* (9-1) *holds, then the proposition* $(\forall x)(\exists \beta)R(x, \beta)$ *is uniformly true, i.e., there is a recursive function* $\boldsymbol{u} : X \to \mathcal{N}$ *such that* $(\forall x)R(x, \boldsymbol{u}(x))$.

Moreover, both claims remain true if we replace B *by* $\mathsf{B} + \mathsf{MP}$.

This theorem establishes a robust connection between provability in $\mathsf{B}^* + \mathsf{MP}$ and effective descriptive set theory.

At the same time, Theorem 9.1 can also be used to establish limitations to what can be proved from our assumptions. For example:

COROLLARY 9.2. (1) $\mathsf{B} + \mathsf{MP} \nvdash (\forall \alpha)[(\forall i)[\alpha(i) = 0] \vee (\exists i)[\alpha(i) > 0]]$.

(2) $\mathsf{B}^* + \mathsf{MP} \nvdash$ "if $A \subseteq X$ and $x \in A \Longrightarrow (\exists s)[x \in N_s \subseteq A]$, then A is $\utilde{\Sigma}^0_1$".

(3) $\mathsf{B}^* + \mathsf{MP}$ *cannot prove that every* $f : X_0 \to \mathbb{N}$ *which is continuous on some subspace* $X_0 \subseteq X$ *has a code as a continuous partial function* $f : X \rightharpoonup \mathbb{N}$.

PROOF. We outline a proof of (2), which is true even for $X = \mathbb{N}$. In this case, every $A \subseteq \mathbb{N}$ is open by the classical definition, provably in B^*, i.e.,

$$\mathsf{B}^* \vdash (\forall n)\Big(n \in A \Longrightarrow (\forall m)[m = n \Longrightarrow m \in A]\Big);$$

and so to prove (2), it is enough to derive a contradiction from the assumption that for every formula $\phi(n)$ in the language of B (with just n free) which defines some $A \subseteq \mathbb{N}$,

$$\mathsf{B} + \mathsf{MP} \vdash (\exists \alpha)(\forall n)[\phi(n) \iff (\exists t)[\alpha(\overline{n}(t)) > 0]].$$

Now if this holds, then (2) of Theorem 9.1 guarantees a recursive α such that

$$n \in A \iff (\exists t)[\alpha(\overline{n}(t)) > 0],$$

which means that A is recursively enumerable and is absurd, since A can be any analytical set.

(3) is proved by a similar argument and (1) is easy. ⊣

9B. Kleene's full Intuitionistic Analysis. The full system I extends B by just one axiom scheme of "continuous choice", *Brouwer's principle for functions* (Kleene and Vesley [1965, ×27.1]) expressed in our notation by

$$(\mathsf{CC}^1_1) \qquad (\forall \alpha)(\exists \beta)A(\alpha, \beta) \Longrightarrow (\exists \sigma)(\forall \alpha)(\exists \beta)[\{\sigma\}^{\mathcal{N},1}(\alpha) = \beta \,\&\, A(\alpha, \beta)].$$

Part (1) of Theorem 9.1 holds with I in place of B, but in Part (2) "uniformly true" must be replaced by "uniformly realizable". The same holds for $\mathsf{I} + \mathsf{MP}$, which (incidentally) establishes the *consistency* of Markov's Principle with I.

Brouwer's principle for functions is a strengthening of *Brouwer's principle for numbers*[31]

$$(\mathsf{CC}^1_0) \qquad (\forall\alpha)(\exists i)A(\alpha,i) \Longrightarrow (\exists\sigma)\forall\alpha)(\exists i)[\{\sigma\}^{\mathcal{N},0}(\alpha) = i \;\&\; A(\alpha,i)],$$

a "choice version" of what is (perhaps) his most famous result, that *every function* $f : \mathcal{N} \to \mathbb{N}$ *is continuous*. It yields strong, positive versions of the independence results in Corollary 9.2, for example

$$\mathsf{I} \vdash \neg(\forall\alpha)[(\forall i)[\alpha(i) = 0] \vee (\exists i)[\alpha(i) > 0]].$$

Veldman [1990], [2008] proves in I several difficult and subtle results which are spectacularly false when read classically, e.g., that the pointset

$$D = \{\alpha \in \mathcal{N} \mid (\forall n)[\alpha(2n) = 0] \vee (\forall n)[\alpha(2n+1) = 0]\}$$

is not $\utilde{\mathbf{\Pi}}^0_1$, in fact not even $\utilde{\mathbf{\Pi}}^1_1$. He also shows that the pointclasses of Borel and projective sets are not closed under negation and that every projective set is $\utilde{\mathbf{\Sigma}}^1_2$. On the positive side, Veldman obtains in these two papers intuitionistic versions of a Borel Hierarchy Theorem and the Separation Theorem 8.1 for analytic sets.

Finally, while B and I can only prove the existence of recursive points $\alpha \in \mathcal{N}$, Kleene and Vesley [1965, Lemma 9.8] show that $\mathsf{B} \vdash \neg(\forall\alpha)\mathrm{GR}(\alpha)$. Even the weaker $(\forall\alpha)\neg\neg\mathrm{GR}(\alpha)$, which is consistent with I, is inconsistent with $\mathsf{B} + \mathsf{MP}$.[32] We do not know Brouwer's opinion of recursive functions, but he certainly objected to MP even though it can be interpreted as expressing the view that *we only have the standard natural numbers*.

Kripke's Schema. To refute MP, Brouwer [1954] used an argument based on the claim that a proposition holds intuitionistically if and only if it is established by a creating subject working in time. Following an unpublished proposal of Kripke, Myhill [1967] rendered Brouwer's premise by the schema

$$(\mathsf{KS}) \qquad (\forall x_1,\dots,x_n)(\exists\alpha)\Big((\exists i)[\alpha(i) = 0] \leftrightarrow \phi\Big)$$

in the language of B, where the free variables in ϕ occur in the list $x_1,\dots,x_n$ (and do not include α). *Kripke's Schema* is (obviously) classically true; it is inconsistent with I; it is consistent with $\mathsf{B} + \mathsf{CC}^1_0$;[33] and $\mathsf{B} + \mathsf{MP} + \mathsf{KS} \vdash \mathsf{LEM}$.

Burgess [1980] proves in $\mathsf{B} + \mathsf{KS}$ a version of Suslin's *Perfect Set Theorem*, that *every uncountable analytic set has a perfect subset*.[34] From our point of view, this is by far the most interesting application of Kripke's Schema because of the analogy with the proof of the classical Separation Theorem 8.1 in $\mathsf{B} + \mathsf{MP}$: they both establish fundamental results of descriptive set theory using intuitionistic logic in classically sound extensions of B which have been defended by some constructive mathematicians.

[31] Kleene and Vesley [1965, *27.2].

[32] J. R. Moschovakis [1971] and [2003].

[33] Krol [1978].

[34] Burgess [1980] also includes an eloquent analysis of the issues that arise in looking for intuitionistic proofs of classical results.

9C. Classically sound semi-constructive systems. There is, however, a very substantial difference between MP and KS, in that the theory $\mathbf{B}$ + KS (easily) does not satisfy (2)—much less (1)—of Theorem 9.1. The fact that $\mathbf{B}$ + MP does, makes it useful for the classical theory, it justifies the extra work needed to give a "constructive" proof of a proposition since—at the least—you get its uniform truth out of it. Burgess' proof does not give any additional, useful information about the Perfect Set Theorem or its many classical proofs and generalizations, cf. Y. N. Moschovakis [2009, 2C.2, 4F.1, 6F.5, 8G.2].[35]

Without more discussion, let us call a formally intuitionistic theory T in the language of $\mathbf{B}$ (or some extension) **semi-constructive** if *it is classically sound and satisfies (1) of Theorem 9.1*, i.e., for $\phi(x, \beta)$ with only x and β free,

$$\text{(9-3)} \quad T \vdash (\forall x)(\exists \beta)\phi(x, \beta) \Longrightarrow T \vdash (\exists \varepsilon)[\mathrm{GR}(\varepsilon) \ \& \ (\forall x)(\exists \beta)[\{\varepsilon\}^{X,1}(x) = \beta \ \& \ \phi(x, \beta)]].$$

$\mathbf{B}$ + MP is such a theory, worth studying, we think; and we also think that it is worth investigating what results other than the Finite Borel Hierarchy Corollary 4.9, the Separation Theorem 8.1, the Suslin-Kleene Theorem (Corollary 8.2) and its Corollary 8.3 can be established in semi-constructive theories—especially theories which have some defensible claim to "constructiveness".

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