The effect of Markov's Principle on the intuitionistic continuum

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Let $\mathbf{M}$ be the minimal two-sorted extension of Heyting Arithmetic, with full induction in the extended language, which was used e.g. by Kleene [1] to formalize the theory of recursive partial functions of type 2. In addition to the defining equations for finitely many primitive recursive function constants, $\mathbf{M}$ has the function existence (or "non-choice") axiom schema

$$\mathrm{AC}_0! : \qquad \forall x \exists! y A(x,y) \to \exists \alpha \forall x A(x, \alpha(x)),$$

but no axiom of countable or dependent choice. Let $\mathbf{T}$ be $\mathbf{M} + \mathrm{BI}_1 + \mathrm{MP}_1$, where BI_1 is Brouwer's principle of bar induction in the form

$$\mathrm{BI}_1 : \ \forall \alpha [\exists x \rho(\overline{\alpha}(x)) = 0 \wedge \forall x (\rho(\overline{\alpha}(x)) = 0 \vee \forall s A(\overline{\alpha}(x) * \langle s \rangle) \to A(\overline{\alpha}(x)))] \to A(\langle\,\rangle)$$

and MP_1 is Markov's Principle in the form

$$\mathrm{MP}_1 : \qquad \forall \alpha [\neg \forall x \neg \alpha(x) = 0 \to \exists x \alpha(x) = 0].$$

Then $\mathbf{T}$ proves:

(i) Every predicate $A(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_m)$ without function quantifiers, indeed every (classically or constructively) Δ^1_1 predicate, is classically decidable with respect to its number variables; that is,

$$\neg\neg \forall x_1 \ldots \forall x_n [A(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_m) \ \vee \ \neg A(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_m)].$$

Hence $\neg\neg \exists \beta \forall x_1 \ldots \forall x_n [\beta(\langle x_0, \ldots, x_n \rangle) = 1 \leftrightarrow A(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_m)]$.

(ii) Every Δ^0_1 predicate has a recursive characteristic function, and the graph of every recursive function is Δ^0_1 (both classically and constructively).

(iii) The constructive arithmetical hierarchy (with or without function parameters) is proper.

Result (i) for arithmetical predicates is due to Robert Solovay (personal communication). A proof of Solovay's result, and proofs of (ii), (iii), and (i) for classically Δ^1_1 predicates, appear in [4] along with other hierarchy results in consistent extensions of intuitionistic analysis. Observe that in $\mathbf{T}$, every constructively Δ^1_1 predicate is also classically Δ^1_1, since MP_1 implies

$$[\exists \alpha \forall x R(\overline{\alpha}(x), z) \leftrightarrow \forall \beta \exists y Q(\overline{\beta}(y), z)] \to [\neg\neg \exists \alpha \forall x R(\overline{\alpha}(x), z) \leftrightarrow \forall \beta \neg\neg \exists y Q(\overline{\beta}(y), z)]$$

if $R(w,z)$ and $Q(v,z)$ are quantifier-free. Results (ii) and (iii) use Kleene's normal form theorem; as an example, we sketch the proof of (iii).

Theorem. $\mathbf{T}$ proves $\Pi^0_n \neq \Delta^0_{n+1} \neq \Sigma^0_{n+1}$ and $\Sigma^0_n \neq \Delta^0_{n+1} \neq \Pi^0_{n+1}$ for $n \in \omega$, so the constructive arithmetical hierarchy (with or without function parameters) is proper.

Proof. Since $\Pi^0_0 = \Sigma^0_0 \neq \Delta^0_1$ by (ii), and $\Pi^0_n \cup \Sigma^0_n \subsetneq \Delta^0_{n+1} = \Sigma^0_{n+1} \cap \Pi^0_{n+1}$, it will suffice to show by induction on n that $\Sigma^0_{n+1} \neq \Delta^0_{n+1}$ and $\Pi^0_{n+1} \neq \Delta^0_{n+1}$.

Basis. $n = 0$. Kleene's normal form theorem, proved in $\mathbf{M}$ (cf. [1]), gives enumerating predicates

$$R_1(x, y, \alpha) \equiv \exists z T(x, y, \overline{\alpha}(z)) \ \text{ and } \ P_1(x, y, \alpha) \equiv \forall z \neg T(x, y, \overline{\alpha}(z))$$

for $\Sigma^0_1(y,\alpha)$ and $\Pi^0_1(y,\alpha)$ respectively, where $T(x,y,w)$ is quantifier-free. **M** proves

$$(*)_1 \qquad \forall\alpha\forall x\forall y[\neg\neg R_1(x,y,\alpha) \leftrightarrow \neg P_1(x,y,\alpha)],$$

so **T** proves that $R_1(x,x,\alpha)$ is not Π^0_1 and $P_1(x,x,\alpha)$ is not Σ^0_1.

Induction Step. By the induction hypothesis with the normal form theorem, there are predicates

$$R_{n+1}(x,y,\alpha) \equiv \exists z C(x,y,z,\alpha) \quad \text{and} \quad P_{n+1}(x,y,\alpha) \equiv \forall z D(x,y,z,\alpha)$$

which enumerate (provably in **M**) $\Sigma^0_{n+1}(y,\alpha)$ and $\Pi^0_{n+1}(y,\alpha)$ respectively, such that **T** proves

$$(*)_n \qquad \forall\alpha\forall x\forall y\forall z[\neg\neg D(x,y,z,\alpha) \leftrightarrow \neg C(x,y,z,\alpha)].$$

Fix α. By (i), **T** proves

$$\neg\neg\exists\zeta\exists\eta\forall x\forall y\forall z[(\zeta((x,y,z))=0 \leftrightarrow C(x,y,z,\alpha))\wedge(\eta((x,y,z))=0 \leftrightarrow D(x,y,z,\alpha))]$$

so $\neg\neg\forall x\forall y\forall z[D(x,y,z,\alpha) \leftrightarrow \neg C(x,y,z,\alpha)]$ by $(*)_n$, and hence

$$(*)_{n+1} \qquad \forall\alpha\forall x\forall y[\neg\neg R_{n+1}(x,y,\alpha) \leftrightarrow \neg P_{n+1}(x,y,\alpha)].$$

Thus $R_{n+1}(x,x,\alpha)$ is not Π^0_{n+1} and $P_{n+1}(x,x,\alpha)$ is not Σ^0_{n+1}.

By [3], Kleene and Vesley's theory **FIM** of intuitionistic analysis (a nonclassical extension of **M** + BI_1 including Brouwer's principle of continuous choice, from which the countable axiom of choice follows) is consistent with $\forall\alpha\neg\neg GR(\alpha)$. Results (i)-(iii) imply that the consistent extension **FIM** + MP_1 of **T** proves $\neg\forall\alpha\neg\neg GR(\alpha)$. Both **T** and **FIM** + MP_1, like other theories considered in [4], satisfy Kleene's recursive instantiation rule: If $\exists\alpha B(\alpha)$ is a closed theorem of the theory, so is $\exists\alpha[GR(\alpha) \wedge B(\alpha)]$ where $GR(\alpha)$ expresses "α is recursive." Thus Markov's Principle increases the classical (but not the constructive) content of the intuitionistic continuum.

Kleene's example in [2], of a recursive fan in which every recursive branch (but not every branch) is finite, shows that the recursive sequences are an inadequate basis for intuitionistic analysis. Markov's Principle helps to explain this fact without implying the constructive existence of nonrecursive sequences. From this point of view, results (i)-(iii) could be considered reasonably strong evidence for Markov's Principle.

References

[1] Kleene, S. C., *Formalized recursive functionals and formalized realizability*, Memoirs, no. 89, American Mathematical Society, 1969.

[2] Kleene, S. C. and Vesley, R. E., *The Foundations of Intuitionistic Mathematics, Especially in Relation to Recursive Functions*, North-Holland, Amsterdam 1965.

[3] Moschovakis, J. R., *Can there be no non-recursive functions*, Journal of Symbolic Logic, Volume 36 (1971), 309-315.

[4] Moschovakis, J. R., *Classical and constructive hierarchies in extended intuitionistic analysis*, Journal of Symbolic Logic, Volume 68 (2003), 1015-1043.