Solutions to M05N Exercises 4.15-4.16

Exercise 4.15. By giving case 7 of the definition, complete the proof that to each formula $E(x_1, \ldots, x_{,})$ of $\mathcal{L}(\mathbf{HA}^{\#})$ with only $x_1, \ldots, x_k$ free there is an almost negative formula $z\,\mathbf{r}\,E(x_1, \ldots, x_k)$ expressing "z realizes $E(\mathbf{x_1}, \ldots, \mathbf{x_k})$." Verify that the resulting formulas are almost negative.

Solution. $z\,\mathbf{r}\,\exists x A(x)$ is $j_1(z)\,\mathbf{r}\,A(j_0(z))$. Since $T(z, x, w)$ is a prime formula of $\mathcal{L}(\mathbf{HA}^{\#})$, the verification that $z\,\mathbf{r}\,E$ is almost negative for every E is completely routine.

Exercise 4.16. Prove that $\mathbf{HA}$ + MP + ECT_0 is (classically) consistent relative to $\mathbf{HA}$.

Solution. We just need to show MP and ECT_0 are realizable, then apply Nelson's Theorem. The proof that MP is realizable will use classical logic (in fact, it will use Markov's Principle on the metalevel). The proof that ECT_0 is realizable will be constructive. Note that *informal*, not formalized, realizability is needed here. Also note that both MP and ECT_0 were stated for $\mathcal{L}(\mathbf{HA})$, not $\mathcal{L}(\mathbf{HA}^{\#})$.

MP is the schema

$$\forall x(A(x) \vee \neg A(x)) \wedge \neg\neg\exists x A(x) \rightarrow \exists x A(x).$$

Claim: $e = \Lambda m\, j(x_0, j_1(\{j_0(m)\}(x_0)))$ realizes MP where $x_0 = \mu x\, j_0(\{j_0(m)\}(x)) \simeq 0$. Assume **(i)** m realizes the hypothesis. Then **(ii)** $j_0(m)$ realizes $\forall x(A(x) \vee \neg A(x))$ and **(iii)** $j_1(m)$ realizes $\neg\neg\exists x A(x)$. By (ii), $\{j_0(m)\}(x)\downarrow$ for each x, and $j_1(\{j_0(m)\}(x))$ realizes $A(\mathbf{x})$ or $\neg A(\mathbf{x})$ depending on whether $j_0(\{j_0(m)\}(x)) = 0$ or $j_0(\{j_0(m)\}(x)) \neq 0$. By (iii) it is impossible that there is no n which realizes $\exists x A(x)$, so it is impossible that $j_0(\{j_0(m)\}(x)) \neq 0$ for all x, so by informal Markov's Principle there is an x such that $j_0(\{j_0(m)\}(x)) = 0$. This proves the claim.

Extended Church's Thesis ECT_0 is the following schema, where $A(x)$ must be almost negative and x, y, w are distinct variables:

$$\forall x[A(x) \rightarrow \exists y B(x, y)] \rightarrow \exists e \forall x[A(x) \rightarrow \exists w \exists y(T(e, x, w) \;\&\; U(w, y) \;\&\; B(x, y))]$$

where for $\mathbf{HA}$, $U(w, y)$ and $T(e, x, w)$ are almost negative formulas.

Claim: $f = \Lambda m\, j(e_0, \Lambda x \Lambda a\, j(w_0, j(y_0, j(j(\varepsilon_T(e_0, x, w_0), \varepsilon_U(w_0, y_0)), j_1(\{\{m\}(x)\}(\varepsilon_A(x)))))))$ realizes ECT_0 where $e_0 = \Lambda x\, j_0(\{\{m\}(x)\}(\varepsilon_A(x)))$, $w_0 = \mu w\, T(e_0, x, w)$ and $y_0 = j_0(\{\{m\}(x)\}(\varepsilon_A(x)))$.

Assume **(i)** m realizes the hypothesis. Then **(ii)** $\{m\}(x)$ realizes $A(\mathbf{x}) \rightarrow \exists y B(\mathbf{x}, y)$ for every x, so **(iii)** if a realizes $A(\mathbf{x})$ then $\{\{m\}(x)\}(a)$ realizes $\exists y B(\mathbf{x}, y)$. By Lemma 4.17, since $A(x)$ is almost negative,

$$\varepsilon_A(x) \text{ is defined and realizes } A(\mathbf{x}) \;\Leftrightarrow\; \text{there is an } a \text{ which realizes } A(\mathbf{x}).$$

Hence **(iv)** if a realizes $A(\mathbf{x})$ then $\{\{m\}(x)\}(\varepsilon_A(x))$ realizes $\exists y B(\mathbf{x}, y)$, so $j_1(\{\{m\}(x)\}(\varepsilon_A(x)))$ realizes $B(\mathbf{x}, \mathbf{y_0})$ where $\mathbf{y_0}$ is the numeral for $y_0 = j_0(\{\{m\}(x)\}(\varepsilon_A(x)))$. The verification that f realizes ECT_0 is straightforward from (i) - (iv) with Lemma 4.17.