Solutions to M05N Exercises 4.12-4.14

Exercise 4.12. Let $j(x,y) = 2^x \cdot 3^y$. Find primitive recursive functions $j_0(z)$ and $j_1(z)$ such that for all $x, y \in \omega$:

$$j_0((x,y)) = x \qquad \text{and} \qquad j_1(j(x,y)) = y.$$

Solution. Exercise 4.5 showed that the predicate $m|n$ ("m divides n") is primitive recursive. Exercise 4.6* showed that the bounded least number operator is primitive recursive. Hence the following are primitive recursive:

$$j_0(z) = \mu m < z\ (2^{m'} \not| z) \qquad \text{and} \qquad j_1(z) = \mu n < z\ (3^{n'} \not| z).$$

To prove that $j_0(j(x,y)) = j_0(2^x \cdot 3^y) = x$, we need only observe that

$$2^m |\ 2^x \cdot 3^y \text{ for all } m \leq x, \text{ but } \quad (2^{x'} \not| 2^x \cdot 3^y).$$

The proof that $j_1(j(x,y)) = j_1(2^x \cdot 3^y) = y$ is similar.

Exercise 4.13. Prove that every closed term of $\mathcal{L}(\mathbf{HA})$ is provably equal, in $\mathbf{HA}$, to a numeral.

Solution. Recall that the *numerals* are defined by: 0 is a numeral, and if $\mathbf{n}$ is a numeral so is $\mathbf{n}'$. The *closed terms* are all the terms which do not contain variables, so they can be defined by: 0 is a closed term, and if s and t are closed terms then so are t', $s+t$ and $s \cdot t$. The proof is by recursion on this inductive definition.

(i) $\vdash_{\mathbf{HA}} 0 = 0$ by Lemma 4.5(a) with $\forall$ elimination.

(ii) $\vdash_{\mathbf{HA}} t = \mathbf{n} \to t' = \mathbf{n}'$ by X17 with the $\forall$ rules, where $\mathbf{n}'$ is the numeral for n' if $\mathbf{n}$ is the numeral for n. If $\vdash_{\mathbf{HA}} t = \mathbf{n}$ then $\vdash_{\mathbf{HA}} t' = \mathbf{n}'$ by R1.

(iii) Suppose s and t are closed terms and $\mathbf{m}$, $\mathbf{n}$ are numerals such that $\vdash_{\mathbf{HA}} s = \mathbf{m}$ and $\vdash_{\mathbf{HA}} t = \mathbf{n}$. Then $\vdash_{\mathbf{HA}} s + t = \mathbf{m} + \mathbf{n}$ by X22, X23, Lemma 4.5(c) and propositional logic. We show by induction on n that $\vdash_{\mathbf{HA}} \mathbf{m} + \mathbf{n} = \mathbf{r}$ where $\mathbf{r}$ is the numeral for $r = m + n$.

Basis. $n = 0$ so $\mathbf{n}$ is 0. Then $\vdash_{\mathbf{HA}} \mathbf{m} + 0 = \mathbf{m}$ by X18.

Induction Step. Assume $\vdash_{\mathbf{HA}} \mathbf{m} + \mathbf{n} = \mathbf{r}$ where $r = m + n$, where $\mathbf{n}$ is the numeral for n and so $\mathbf{n}'$ is the numeral for n'. Then

a. $\vdash_{\mathbf{HA}} \mathbf{m} + \mathbf{n}' = (\mathbf{m} + \mathbf{n})'$ by X19.

b. $\vdash_{\mathbf{HA}} (\mathbf{m} + \mathbf{n})' = \mathbf{r}'$ by the induction hypothesis with X17 and R1.

c. $\vdash_{\mathbf{HA}} \mathbf{m} + \mathbf{n}' = \mathbf{r}'$ by a and b with Lemma 4.5(c) and propositional logic, where $\mathbf{r}'$ is the numeral for $r' = m + n'$.

(iv) If $\vdash_{\mathbf{HA}} s = \mathbf{m}$ and $\vdash_{\mathbf{HA}} t = \mathbf{n}$ then $\vdash_{\mathbf{HA}} s \cdot t = \mathbf{m} \cdot \mathbf{n}$ by X24, X25, Lemma 4.5(c) and propositional logic. Now prove by induction on n that $\vdash_{\mathbf{HA}} \mathbf{m} \cdot \mathbf{n} = \mathbf{r}$ where $\mathbf{r}$ is the numeral for $r = m \cdot n$, just as for (iii) but using X20 and X21 instead of X18 and X19 respectively.

Exercise 4.14. (briefly) (a) 0 realizes C because no number realizes the hypothesis of C. (b) The hypothesis of C is provable in $\mathbf{PA}$. Since the characteristic function of the predicate $\{x\}(x)\downarrow$ is not recursive, the conclusion of C is classically false and even inconsistent with $\mathbf{HA}$ (hence also with $\mathbf{PA}$).

We need the fact that $\vdash_{\mathbf{HA}} \forall e \forall x \forall w [T(e,x,w) \to \exists y U(w,y)]$, as well as (i) - (iv) of the remark in the notes preceding this exercise. We also need

$$(*) \qquad \vdash_{\mathbf{HA}} \forall e \exists f \forall x\ [\exists w T(f,x,w) \leftrightarrow \exists w [T(e,x,w)\ \&\ \neg U(w,0)].$$

Now assume for contradiction, from the conclusion of C, that e satisfies

$$\forall x \exists w \exists y [T(e,x,w)\ \&\ U(w,y)\ \&\ (y = 0 \to \exists z T(x,x,z))\ \&\ (y \neq 0 \to \forall z \neg T(x,x,z))].$$

Then $\forall x [\forall z \neg T(x,x,z) \leftrightarrow \exists w [T(e,x,w)\ \&\ \neg U(w,0)]]$ and so by $(*)$ we may assume there is an f so that $\forall x [\exists w T(f,x,w) \leftrightarrow \forall z \neg T(x,x,z)]$. But then $\exists w T(f,f,w) \leftrightarrow \forall z \neg T(f,f,z)$, which is impossible.