Solutions to M05N Exercises 4.1(a)(c) and 4.3

Exercise 4.1. Prove Lemma 4.2(a) and (c). ((b) was proved in class.)

Solutions. First we derive for **Pd**[=] the rule of $\forall$ introduction:

$$\text{From} \quad A(x) \quad \text{conclude} \quad \forall x A(x),$$

where $A(x)$ does not depend on any assumption formula in which x occurs free. Let C be the formula $b = b \to (b = b \to b = b)$ where b is a variable not occurring in the given **Pd**[=] deduction $\mathcal{D}$ of $A(x)$; say $A(x)$ is the n^{th} line of that deduction. Extend $\mathcal{D}$ as follows:

n. $A(x)$. [deduced from assumptions with no free x]

n+1. $A(x) \to (C \to A(x))$. [axiom by X1]

n+2. $C \to A(x)$. [from $n, n+1$ by R1]

n+3. $C \to \forall x A(x)$. [from $n+2$ by R2, without varying x]

n+4. C. [axiom by X1]

n+5. $\forall x A(x)$. [from $n+3$ and $n+4$ by R1]

The advantage of this rule for **Pd**[=] (and **HA**) is that it gives us the universal closure of every axiom which is a particular formula rather than a schema. Another short argument using X1, R1 and X11 shows that **Pd**[=] and **HA** are closed under the rule of $\forall$ elimination:

$$\text{From} \quad \forall A(x) \quad \text{conclude} \quad A(t),$$

where t is any term free for x in $A(x)$.

(a) To prove $\forall x(x = x)$ in **Pd**[=], use $\forall$I twice, as follows.

1. $a = a$. [axiom by XE1]
2. $\forall a(a = a)$. [from 1 by $\forall$I]
3. $\forall a(a = a) \to x = x$. [X11]
4. $x = x$. [from 2 and 3 by R1]
5. $\forall x(x = x)$. [from 4 by $\forall$I]

(c) To prove $\forall x \forall y \forall z(x = y \; \& \; y = z \to x = z)$ in **Pd**[=], first use XE2 with $\forall$I and $\forall$E to prove that **Pd**[=] $\vdash (y = x \to (y = z \to x = z))$:

1. $a = b \to (a = c \to b = c)$. [axiom by XE2 with $z = c$ as the $P(z)$]
2. $\forall a \forall b \forall c(a = b \to (a = c \to b = c))$. [from 1 by repeated $\forall$I]
3. $y = x \to (y = z \to x = z)$. [from 2 by repeated $\forall$E]

Next observe that **Pd**[=] $\vdash x = y \to y = x$ by $\forall$E from Lemma 4.2(b), so by propositional logic: **Pd**[=] $\vdash x = y \to (y = z \to x = z)$, and so **Pd**[=] $\vdash x = y \; \& \; y = z \to x = z$. Three uses of $\forall$I complete the proof.

Exercise 4.3. Prove Corollary 4.4(a) and outline the proof of (b), treating completely the inductive cases for $\forall$ and $\exists$.

(a) $\vdash_{\mathbf{Pd}[=]} \forall x \forall y(x = y \rightarrow t(x) = t(y))$ if x, y are variables free for z in the term $t(z)$.

Proof. Recall that the terms of $\mathbf{Pd}[=]$ are individual variables, so $t(z)$ may be z, x, y or another variable $w \neq x, y, z$. We take cases:

(i) $t(z)$ is z, where $z \neq x, y$. Then $\vdash_{\mathbf{Pd}[=]} \forall x \forall y(x = y \rightarrow x = y)$ by $\forall$I, using the fact that $\vdash_{\mathbf{Pd}[=]} A \rightarrow A$ for every formula A of the language with $=$.

(ii) $t(z)$ is x. Then $\vdash_{\mathbf{Pd}[=]} \forall x \forall y(x = y \rightarrow x = x)$ using Lemma 4.2(a), X1, R1 and $\forall$E and $\forall$I.

(iii) $t(z)$ is y. Similar.

(iv) $t(z)$ is w where $w \neq x, y, z$. Then $\vdash_{\mathbf{Pd}[=]} \forall x \forall y(x = y \rightarrow w = w)$ by Lemma 4.2(a), X1, R1 and the $\forall$ rules.

(b) $\vdash_{\mathbf{Pd}[=]} \forall x \forall y(x = y \rightarrow (A(x) \leftrightarrow A(y)))$ where $A(z)$ is a formula of $\mathcal{L}(\mathbf{Pd}[=])$ and x, y are distinct variables free for z in $A(z)$.

Proof is by induction on the logical form of $A(z)$. First observe that XE2 can be strengthened to $a = b \rightarrow (P(a) \leftrightarrow P(b))$ using $\forall$I and $\forall$E with Lemma 4.2(b) and propositional logic. Then $\forall x \forall y(x = y \rightarrow (P(x) \leftrightarrow P(y))$ follows using the $\forall$ rules, taking extra care with the quantifiers if x is b or y is a. The cases for $\&$, $\vee$, $\rightarrow$ and $\neg$ are straightforward, e.g. if $A(z)$ is $B(z) \vee C(z)$ where the induction hypothesis holds for $B(z)$ and $C(z)$ then we just need to check that $(B(x) \leftrightarrow B(y))$, $(C(x) \leftrightarrow C(y)) \vdash_{\mathbf{Pd}[=]} (A(x) \leftrightarrow A(y))$, a matter of propositional logic. Then use R1, the Deduction Theorem for $\mathbf{Pd}[=]$, the transitivity of $\vdash_{\mathbf{Pd}[=]}$ and the $\forall$ rules.

If $A(z)$ is $\forall v B(v, z)$ where $v \neq x, y, z$, and if $\vdash_{\mathbf{Pd}[=]} \forall x \forall y(x = y \rightarrow (B(v, x) \leftrightarrow B(v, y)))$ by the induction hypothesis, then

1. $x = y \vdash_{\mathbf{Pd}[=]} (B(v, x) \leftrightarrow B(v, y)))$. [from the induction hypothesis by R1]
2. $(B(v, x) \leftrightarrow B(v, y))) \vdash_{\mathbf{Pd}[=]} (B(v, x) \rightarrow B(v, y)))$. [by X4 with R1]
3. $(B(v, x) \leftrightarrow B(v, y))), B(v, x) \vdash_{\mathbf{Pd}[=]} B(v, y)$. [from 2 by R1]
4. $x = y, B(v, x) \vdash_{\mathbf{Pd}[=]} B(v, y)$. [from 1,2,3 by transitivity of $\vdash_{\mathbf{Pd}[=]}$]
5. $\forall v B(v, x) \vdash_{\mathbf{Pd}[=]} B(v, x)$. [by R1 from X11]
6. $x = y, \forall v B(v, x) \vdash_{\mathbf{Pd}[=]} B(v, y)$. [from 4,5 by transitivity of $\vdash_{\mathbf{Pd}[=]}$]
7. $x = y, \forall v B(v, x) \vdash_{\mathbf{Pd}[=]} \forall v B(v, y)$. [from 6 by $\forall$I]
8. $x = y \vdash_{\mathbf{Pd}[=]} \forall v B(v, x) \rightarrow \forall v B(v, y)$. [from 7 by the Deduction Theorem]

A similar argument (using X5 instead of X4) shows that $x = y \vdash_{\mathbf{Pd}[=]} \forall v B(v, y) \rightarrow \forall v B(v, x)$. Now use the Deduction Theorem, X3, R1 and $\forall$I to finish this case.

If $A(z)$ is $\exists v B(v, z)$ where $v \neq x, y, z$, and if $\vdash_{\mathbf{Pd}[=]} \forall x \forall y(x = y \rightarrow (B(v, x) \leftrightarrow B(v, y)))$ by the induction hypothesis, just replace lines 5 - 8 of the previous argument by

5′. $B(v, y) \vdash_{\mathbf{Pd}[=]} \exists v B(v, y)$. [by R1 from X12]

6′. $x = y, B(v, x) \vdash_{\mathbf{Pd}[=]} \exists v B(v, y)$. [from 4, 5′ by transitivity of $\vdash_{\mathbf{Pd}[=]}$]

7′. $x = y \vdash_{\mathbf{Pd}[=]} B(v, x) \rightarrow \exists v B(v, y)$. [from 6′ by the Deduction Theorem]

8′. $x = y \vdash_{\mathbf{Pd}[=]} \exists v B(v, x) \rightarrow \exists v B(v, y)$. [from 7′ by R3]

and complete the proof as for the case of $\forall$.