A Logical Look at Kripke's Idea of Choice Sequence

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Saul Kripke ended his lecture *"Free choice sequences: A temporal interpretation compatible with acceptance of classical mathematics"* at the symposium "L. E. J. Brouwer: 50 years later" in Amsterdam, December 9, 2016 with a question and an answer:

"So, what am I arguing?

"That a Brouwerian theory of free choice sequences could be added to classical mathematics without any constructive doubts as to its validity."

Before considering to what extent Kripke's idea is *logically* feasible, we need to understand *what it is about*.

Infinite sequences of natural numbers

The natural numbers *n* are generated by starting with 0 and repeating the operation of taking the successor. The classical mathematician puts them all in a set $\mathbb{N} = \{0, 1, 2, \ldots\}$. For the intuitionist $\mathbb{N}$ represents an incomplete process, but they both agree that these are the (standard) natural numbers and that $\forall \mathrm{m} \forall \mathrm{n}(\mathrm{m} = \mathrm{n} \vee \neg(\mathrm{m} = \mathrm{n}))$ holds.

An infinite sequence of natural numbers can be thought of as (the sequence of values of) a function α from $\mathbb{N}$ to $\mathbb{N}$:

$$\alpha(0), \alpha(1), \alpha(2), \alpha(3), \ldots$$

Rational numbers can be coded by natural numbers, and real numbers can be represented by Cauchy sequences of rationals, so mathematical analysis (both classical and intuitionistic) reduces to the theory of infinite sequences of natural numbers. But the intuitionist does not accept $\forall \alpha \forall \beta [\alpha = \beta \vee \neg(\alpha = \beta)]$.

Intuitionistic logic needs $\neg$, &, $\vee$, $\rightarrow$, $\forall$ and $\exists$

For an intuitionist, $\neg A$ means "I can derive a contradiction from any 'proof' of A." So $A \rightarrow \neg\neg A$ but not $\neg\neg A \rightarrow A$.

For an intuitionist, $A \vee B$ means "Either I can verify A, or I can verify B." So $\neg A \vee \neg B \rightarrow \neg(A \,\&\, B)$, but not conversely.

For an intuitionist, $\exists x A(x)$ means "I can find an object x and verify $A(x)$." So $\exists x \neg A(x) \rightarrow \neg\forall x A(x)$, but not conversely.

For an intuitionist, $\neg\forall x \neg A(x)$ is equivalent to $\neg\neg\exists x A(x)$, but not to $\exists x A(x)$ or $\exists x \neg\neg A(x)$.

Classical logic does not need $\vee$ or $\exists$ because classically

- $\neg\neg A \leftrightarrow A$,
- $\neg(\neg A \,\&\, \neg B) \leftrightarrow A \vee B$,
- and $\neg\forall x \neg A(x) \leftrightarrow \exists x A(x)$.

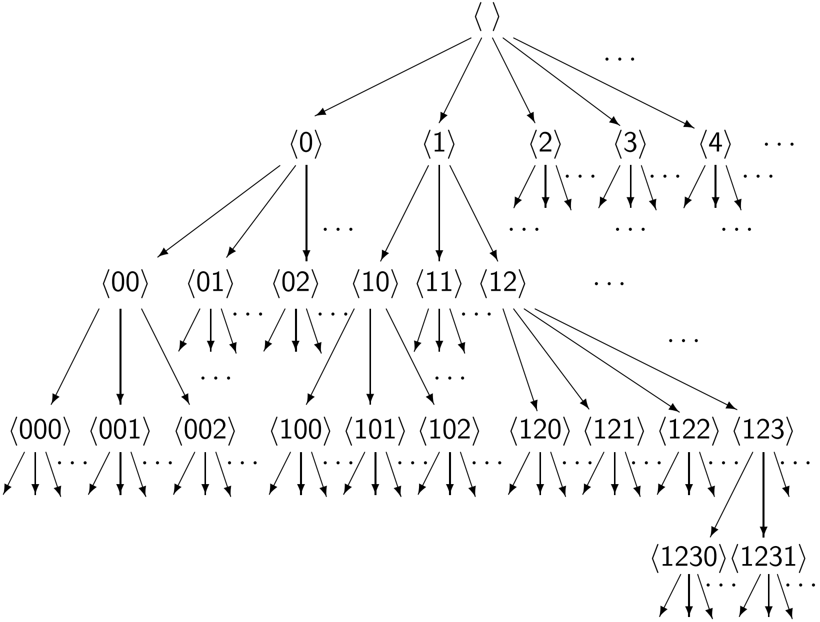
Brouwer's idea of free choice sequences

For Brouwer, some sequences are more definite than others.
The values of a *lawlike* sequence are completely determined, e.g. $\pi(n) =$ the $(n+1)$st digit in the decimal expansion of π.
In his 1907 dissertation Brouwer accepted Cantor's proof that there are more infinite sequences than definitions or laws, so Brouwer's *reduced continuum of lawlike reals* is incomplete.

Some years later he introduced the concept of a *free choice sequence of natural numbers*, belonging to a *spread* which might (or might not) impose bounds on successive choices.

Brouwer's *full continuum* includes real numbers representable by choice sequences which may be only *potentially* infinite. Brouwer's *universal spread* consists of *all choice sequences of natural numbers in the process of generation*. To a classical mathematician Brouwer's universal spread is just Baire Space.

The Universal Spread:



Brouwer's view led him to accept in principle

- Countable Choice AC_{01}:
  $\forall n \exists \alpha A(n, \alpha) \rightarrow \exists \beta \forall n A(n, \lambda m.\beta(\langle n, m \rangle))$.
- Continuous Choice: If $\forall \alpha \exists n A(\alpha, n)$, there is a (code $\{\sigma\}$ of a) *continuous* function so that $\forall \alpha A(\alpha, \{\sigma\}(\alpha))$ holds.
- "The Bar Theorem": If $B(w)$ codes a "thin bar" on the universal spread (so for each α there is exactly one initial segment $\overline{\alpha}(n) = \langle \alpha(0), \ldots, \alpha(n-1) \rangle$ such that $B(\overline{\alpha}(n))$ holds), and if $A(w)$ is a property of nodes w so that
  - $\forall w(B(w) \rightarrow A(w))$, and
  - "*A propagates back across the nodes*": if $A(u)$ holds of every immediate successor u of w then $A(w)$ holds also,

  then $A(w)$ holds for all nodes above the bar including $\langle\,\rangle$.

Note: The Bar Theorem is derivable from AC_{01} using classical logic, but continuous choice is *false* for classical Baire space.

Is Brouwer's view consistent?

Using continuous choice and bar induction with intuitionistic logic, Brouwer proved in 1924 that every total function on the unit continuum is uniformly continuous. With classical logic there are counterexamples. The consistency question led to formalizing Brouwer's analysis. Heyting made a brave attempt in 1930. Kleene and Vesley were more successful in 1965.

- In a two-sorted language Kleene axiomatized the common part **B** of intuitionistic and classical analysis, with intuitionistic logic, countable choice and bar induction.
- Kleene extended **B** to **I** by adding a continuous choice principle stronger than Brouwer's, and proved by function-realizability that **I** is consistent relative to **B**.
- Vesley and Kleene formalized much of Brouwer's real analysis in **I**, establishing its faithfulness to his view.

Strong counterexamples and the creating subject

Some of Brouwer's strong counterexamples to classical principles are provable using continuity arguments, e.g. $\neg\forall\alpha(\forall \mathrm{x}\alpha(\mathrm{x}) = 0 \vee \neg\forall \mathrm{x}\alpha(\mathrm{x}) = 0)$ is provable in **I**.

Markov's Principle MP (which Brouwer did not accept) is consistent with but unprovable in **I**, as Kleene showed.

To refute it and similar principles Brouwer used another argument. He invented a creating subject (c.s.) working in stages indexed by the natural numbers $n \in \mathbb{N}$.

- At each stage n the c.s. tries to verify a property A.
- At each stage n the c.s. is aware whether or not (s)he has verified A.
- If the c.s. verifies A at some stage n then A holds.
- If it is impossible for the c.s. to verify A at any stage n then $\neg A$ holds.

Kripke's schemas, strong and weak

More than fifty years ago Kripke proposed using a binary free choice sequence α to track the creating subject's efforts to decide an assertion A (not involving α).

- As long as the creating subject has not verified A by time n, (s)he continues to set $\alpha(n) = 0$.
- If the c.s. has verified A by time n, (s)he sets $\alpha(n) = 1$.

Kripke's Schema KS asserts that A holds *if and only if* the c.s. verifies A at some time n:

$\mathrm{KS}:\ \exists\alpha(\exists \mathrm{x}\alpha(\mathrm{x}) \neq 0 \leftrightarrow \mathrm{A})$, with α not free in A.

Kripke, Myhill and Troelstra prefer *Weak Kripke's Schema*:

$\mathrm{WKS}: \exists\alpha[(\exists \mathrm{x}\alpha(\mathrm{x}) \neq 0 \rightarrow \mathrm{A})\,\&\,(\forall \mathrm{x}\alpha(\mathrm{x}) = 0 \rightarrow \neg\mathrm{A})]$

where α is not free in A. The second conjunct in WKS is intuitionistically equivalent to $(\mathrm{A} \rightarrow \neg\neg\exists \mathrm{x}\alpha(\mathrm{x}) \neq 0)$.

WKS for *sentences* A is consistent with Kleene's intuitionistic analysis **I**, by a classical realizability argument with α constant ($\alpha(n) = 1$ if A is realized by some function σ, 0 otherwise).

But WKS needs a free sequence variable in the A to refute Markov's principle MP_1: $\forall\beta(\neg\neg\exists \mathrm{x}\beta(\mathrm{x}) = 0 \rightarrow \exists \mathrm{x}\beta(\mathrm{x}) = 0)$.

Myhill observed that WKS with free sequence variables conflicts with Kleene's strong continuous choice principle.

Krol sidestepped the conflict by weakening the hypothesis $\forall\alpha\exists\beta\mathrm{A}(\alpha, \beta)$ of Kleene's continuous choice axiom to $\forall\alpha\exists\mathrm{x}\mathrm{A}(\alpha, \mathrm{x})$, and similarly weakening the conclusion.

Vesley proposed an alternative: weaken KS to *Vesley's Schema* VS, which entails the creating subject counterexamples and is special realizable, hence consistent with **I**.

Kripke's idea of free choice sequence

Kripke now suggests viewing Brouwer's choice sequences as a *supplement* to classical mathematics rather than an alternative. Intuitionistic logic is appropriate for free choice sequences, which are constructed in time, "because there is no 'end of time' when everything about them is determined."

Consider an immortal mathematician $\mathcal{M}$ who understands classical Baire space $\mathbb{N}^{\mathbb{N}}$. Confronted at time 0 with an as yet unsolved mathematical problem ψ, $\mathcal{M}$ begins trying to solve ψ *and* constructing a binary choice sequence α_ψ, with $\alpha_\psi(n) = 0$ if $\mathcal{M}$ has not solved ψ by time n, otherwise $\alpha_\psi(n) = 1$.

Then $\forall n(\alpha_\psi(n) = 0 \vee \alpha_\psi(n) = 1)$, but until ψ has been solved *or proved to be unsolvable* not all the values of α_ψ will be known. If b ranges over all binary sequences, $\mathcal{M}$ can assert $\neg\neg\exists b \forall n(\alpha_\psi(n) = b(n))$ but not always $\exists b \forall n(\alpha_\psi(n) = b(n))$.

Kripke said he had in mind a branching time Beth (rather than Kripke) model, satisfying WKS, for his free choice sequences.

In 1970 Kreisel and Troelstra developed a 3-sorted formal system for intuitionistic analysis, with lawlike sequence variables $\mathrm{e}, \mathrm{a}, \mathrm{b}, \ldots$ and choice sequence variables $\alpha, \beta, \ldots$, and an axiom of analytic data enabling the elimination of the choice sequences variables. **CS** proves $\forall\alpha\neg\neg\exists\mathrm{b}(\alpha = \mathrm{b})$ and $\neg\forall\alpha\exists\mathrm{b}(\alpha = \mathrm{b})$. The choice sequence part of **CS** turned out to be equivalent to an inessential extension of Kleene's **I**.

IC is another 3-sorted formal system, with choice sequence variables $\alpha, \beta, \ldots$ and definite sequence variables $\mathrm{a}, \mathrm{b}, \ldots$, extending Kleene's **I** by adding a faithful translation $\mathbf{C}^\circ$ of classical analysis $\mathbf{C} = \mathbf{B} + (\neg\neg\mathrm{A} \to \mathrm{A})$. $\forall\alpha\neg\neg\exists\mathrm{b}(\alpha = \mathrm{b})$ is an axiom of **IC**, but $\neg\forall\alpha\exists\mathrm{b}(\alpha = \mathrm{b})$ is *independent of* **IC** assuming classical analysis has a proper ω-model $\mathcal{M} = (\omega, \mathcal{C})$ with $\mathcal{C} \neq \omega^\omega$.

The problem of reasoning constructively and classically in the same language is easily solved. Classical logic can be expressed *negatively* (without $\vee$ and $\exists$) using the inductively defined Gödel-Gentzen negative translation $\mathrm{A} \mapsto \mathrm{A}^\circ$:

1. $\mathrm{P}^\circ \equiv \neg\neg\mathrm{P}$ if P is a proposition letter; $(\mathrm{s} = \mathrm{t})^\circ \equiv (\mathrm{s} = \mathrm{t})$.
2. $(\mathrm{A} \,\&\, \mathrm{B})^\circ \equiv (\mathrm{A}^\circ \,\&\, \mathrm{B}^\circ)$.
3. $(\mathrm{A} \vee \mathrm{B})^\circ \equiv \neg(\neg\mathrm{A}^\circ \,\&\, \neg\mathrm{B}^\circ)$.
4. $(\mathrm{A} \rightarrow \mathrm{B})^\circ \equiv (\mathrm{A}^\circ \rightarrow \mathrm{B}^\circ)$.
5. $(\neg\mathrm{A})^\circ \equiv \neg\mathrm{A}^\circ$.
6. $(\forall \mathrm{x} \mathrm{A}(\mathrm{x}))^\circ \equiv \forall \mathrm{x} \mathrm{A}^\circ(\mathrm{x})$.
7. $(\exists \mathrm{x} \mathrm{A}(\mathrm{x}))^\circ \equiv \neg\forall \mathrm{x} \neg\mathrm{A}^\circ(\mathrm{x})$.

Intuitionistic logic proves $\neg\neg\mathrm{A}^\circ \rightarrow \mathrm{A}^\circ$ for all formulas A.

Write $\mathrm{A} \overset{\circ}{\vee} \mathrm{B}$ for $\neg(\neg\mathrm{A} \,\&\, \neg\mathrm{B})$, and $\exists^\circ \mathrm{x} \mathrm{A}(\mathrm{x})$ for $\neg\forall \mathrm{x} \neg\mathrm{A}(\mathrm{x})$. Then $\vee$ and $\exists$ retain their constructive interpretations.

C$^\circ$ is a *negative* version of classical analysis.

The language $\mathcal{L}(\mathbf{C}^\circ)$ has variables $\mathrm{i}, \mathrm{j}, \ldots, \mathrm{q}, \mathrm{w}, \mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{i}_1, \ldots$ over natural numbers and $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{a}_1, \ldots$ over sequences of numbers; constants for $0, ', +, \cdot$ and additional primitive recursive functions as needed; Church's λ; equality $=$ for numbers; parentheses, also denoting function application; and the logical constants $\&, \neg, \rightarrow, \forall$.

C$^\circ$-*terms* (type 0) and C$^\circ$-*functors* (type 1) are defined inductively. Number variables and 0 are C$^\circ$-*terms*, sequence variables and $'$ are C$^\circ$-*functors*, additional C$^\circ$-*terms* are defined by application, and if t is a C$^\circ$-*term* then $\lambda \mathrm{x}.\mathrm{t}$ is a C$^\circ$-*functor*.

If s, t are C$^\circ$-*terms* then $(\mathrm{s} = \mathrm{t})$ is a *prime* C$^\circ$-*formula*. If A, B are C$^\circ$-*formulas* so are $(\mathrm{A} \,\&\, \mathrm{B}),\ (\mathrm{A} \rightarrow \mathrm{B}),\ (\neg \mathrm{A}),\ (\forall \mathrm{x} \mathrm{A}), (\forall \mathrm{b} \mathrm{A})$.

All the logical rules and axioms for $\&, \neg, \rightarrow, \forall \mathrm{x}, \forall \mathrm{b}$ *are intuitionistic*. The postulates for $\overset{\circ}{\vee}$ and $\exists^\circ$ are derivable.

Mathematical axioms of $\mathbf{C}^\circ$:

- $=$ is an equivalence relation, $\mathrm{x} = \mathrm{y} \rightarrow \mathrm{a}(\mathrm{x}) = \mathrm{a}(\mathrm{y})$, 0 is not a successor, and $'$ is one-to-one.
- Primitive recursive definitions of function constants.
- Mathematical induction: $\mathrm{A}(0) \,\&\, \forall \mathrm{x}(\mathrm{A}(\mathrm{x}) \rightarrow \mathrm{A}(\mathrm{x}')) \rightarrow \mathrm{A}(\mathrm{x})$ for C$^\circ$-formulas $\mathrm{A}(\mathrm{x})$.
- λ-reduction: $(\lambda \mathrm{x}.\mathrm{r}(\mathrm{x}))(\mathrm{t}) = \mathrm{r}(\mathrm{t})$ for C$^\circ$-terms $\mathrm{r}(\mathrm{x}), \mathrm{t}$.
- Negative axiom of countable choice for C$^\circ$-formulas A:
  $\mathrm{AC}_{01}^{\mathbf{C}^\circ} : \ \forall \mathrm{x} \exists^\circ \mathrm{a} \mathrm{A}(\mathrm{x}, \mathrm{a}) \rightarrow \exists^\circ \mathrm{b} \forall \mathrm{x} \mathrm{A}(\mathrm{x}, \lambda \mathrm{y}.\mathrm{b}(2^{\mathrm{x}} \cdot 3^{\mathrm{y}}))$.

Proposition. $\mathbf{C}^\circ \vdash \neg\neg \mathrm{A} \rightarrow \mathrm{A}$ for formulas A of $\mathcal{L}(\mathbf{C}^\circ)$.

Proposition. If $\mathbf{B}^-$ is the subsystem of $\mathbf{I}$ with AC_{01} but without bar induction or continuous choice, then

- $\mathbf{C} = \mathbf{B}^- + (\neg\neg \mathrm{A} \rightarrow \mathrm{A})$ proves bar induction.
- $\mathrm{A} \mapsto \mathrm{A}^\circ$ is a faithful translation of $\mathbf{C}$ onto $\mathbf{C}^\circ$.

IC is a 3-sorted extension of **I** and $\mathbf{C}^\circ$, with added existential quantifiers $\exists \mathrm{b}$ over definite (classical) sequences, with intuitionistic logic throughout, and an *end of time* axiom

$$\text{ET:} \quad \forall \alpha \neg\neg \exists \mathrm{b} \forall \mathrm{x}\, \alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$$

(abbreviated $\forall \alpha \exists^\circ \mathrm{b} \forall \mathrm{x} \alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$).

I and $\mathbf{C}^\circ$ have the same primitive recursive function constants. Both sorts of sequence variables are *functors*. Terms, functors and formulas without choice sequence variables are C-*terms*, C-*functors* and C-*formulas* respectively.

The logical axioms and rules of **IC** are those of **I** extended to $\mathcal{L}(\mathbf{IC})$, plus intuitionistic postulates for $\forall \mathrm{b}$, $\exists \mathrm{b}$ (e.g. $\mathrm{A}(\mathrm{u}) \to \mathrm{B} \; / \; \exists \mathrm{b} \mathrm{A}(\mathrm{b}) \to \mathrm{B}$, where u is a C-functor free for b in $\mathrm{A}(\mathrm{b})$ and b is not free in B). The mathematical axioms of **IC** are those of **I** extended to $\mathcal{L}(\mathbf{IC})$, plus $\mathsf{AC}_{01}^{\mathbf{C}^\circ}$ (only for negative C-formulas, as in $\mathbf{C}^\circ$), plus ET.

Proposition. $\mathbf{IC} \vdash \forall \mathrm{b} \exists \alpha \forall \mathrm{x} \mathrm{b}(\mathrm{x}) = \alpha(\mathrm{x})$.

Now assume $\mathcal{M} = (\omega, \mathcal{C})$ *is a classical* ω*-model of* **C**, so $\mathcal{C}$ is recursively closed.

Using $\mathcal{M}$ we can define a ${}^{\mathcal{C}}$*realizability interpretation*, using elements of $\mathcal{C}$ as the *actual* ${}^{\mathcal{C}}$realizing objects and to interpret free sequence variables of both sorts. *Potential* ${}^{\mathcal{C}}$realizing objects, and interpretations of free *choice sequence* variables in the corresponding definition of *agreement*, are elements of ω^ω.
A *sentence* of $\mathcal{L}(\mathbf{IC})$ is ${}^{\mathcal{C}}$*realizable* if and only if it has a *recursive* ${}^{\mathcal{C}}$realizer, and a formula is ${}^{\mathcal{C}}$*realizable* if its universal closure is.

Lemma. For every negative C-formula E of $\mathcal{L}(\mathbf{IC})$ with only Ψ free there is a primitive recursive potential ${}^{\mathcal{C}}$realizer τ_{E} for E such that for each interpretation $\mathit{\Psi}$ of Ψ by elements of $\mathcal{C} \cup \omega$:

1. If E is ${}^{\mathcal{C}}$realized-$\mathit{\Psi}$ by some $\varepsilon \in \mathcal{C}$ then E is true-$\mathit{\Psi}$ in $\mathcal{M}$.
2. If E is true-$\mathit{\Psi}$ in $\mathcal{M}$ then τ_{E} ${}^{\mathcal{C}}$realizes-$\mathit{\Psi}$ E.

Cor. A sentence E of $\mathcal{L}(\mathbf{C}^\circ)$ is ${}^{\mathcal{C}}$realizable iff E is true in $\mathcal{M}$.

Theorem. If $\mathrm{F}_1, \ldots, \mathrm{F}_\mathrm{n}, \mathrm{E}$ are formulas of $\mathcal{L}(\textbf{IC})$ such that $\mathrm{F}_1, \ldots, \mathrm{F}_\mathrm{n} \vdash_{\textbf{IC}} \mathrm{E}$ and $\mathrm{F}_1, \ldots, \mathrm{F}_\mathrm{n}$ are all ${}^{\mathcal{C}}$realizable, then E is ${}^{\mathcal{C}}$realizable. Since $0 = 1$ is not ${}^{\mathcal{C}}$realizable, **IC** is consistent.

Corollary. **IC** $+$ NegTh($\mathcal{M}$) is consistent, where NegTh($\mathcal{M}$) is the set of all sentences of $\mathcal{L}(\textbf{C}^{\circ})$ which are true in $\mathcal{M}$.

Theorem. $\forall\alpha\exists\mathrm{b}\forall\mathrm{x}\,\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$ is independent of **IC**, assuming **C** has a proper ω-model.

Proof. If $\mathcal{C} = \omega^\omega$ then $\forall\alpha\exists\mathrm{b}\forall\mathrm{x}\,\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$ is ${}^{\mathcal{C}}$realizable, so **IC** $\nvdash \neg\forall\alpha\exists\mathrm{b}\forall\mathrm{x}\,\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$, so **IC** $+ \forall\alpha\exists\mathrm{b}\forall\mathrm{x}\,\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$ is consistent. And if $\mathcal{C} \neq \omega^\omega$ then $\neg\forall\alpha\exists\mathrm{b}\forall\mathrm{x}\,\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$ is ${}^{\mathcal{C}}$realizable, so **IC** $+ \neg\forall\alpha\exists\mathrm{b}\forall\mathrm{x}\,\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$ is also consistent.

Corollary. If $\mathcal{M}$ is a proper ω-model of **C**, then **IC** $+ \neg\forall\alpha\exists\mathrm{b}\forall\mathrm{x}\,\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x}) +$ NegTh($\mathcal{M}$) is consistent.

By relativizing to ${}^{\mathcal{C}}$realizability/$\mathcal{C}$ these results remain true when all classically true sentences of arithmetic are added.

Markov's Principle MP_1: $\neg\neg\exists x\alpha(x) = 0 \rightarrow \exists x\alpha(x) = 0$ is refuted by Vesley's Schema VS, which is ${}^{\mathcal{C}}$realizable and hence consistent with **IC**. Classically, a weaker version of WKS:

$$\text{WWKS:}\quad \exists\beta[\forall x\beta(x) = 0 \leftrightarrow \neg A]$$

is ${}^{\mathcal{C}}$realizable (so consistent) for *sentences* A of $\mathcal{L}(\textbf{IC})$.

Interpretation: Even if the creating subject (assuming just the principles of **IC**) can prove all classically true arithmetical sentences (including Markov's Principle for *recursive* sequences) and all true negative sentences about definite (classical or lawlike) sequences,

- by ET, the creating subject cannot construct a choice sequence which differs from every definite sequence;
- the creating subject will be unable to decide if every choice sequence is extensionally equal to a definite sequence or not.

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