MARKOV'S PRINCIPLE, MARKOV'S RULE AND THE NOTION OF CONSTRUCTIVE PROOF

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1. Introduction

In [45] and again in [46] Georg Kreisel reflected at length on *Church's Thesis* CT, the principle formulated in 1936 by Alonzo Church ([10]) as a definition: "We now define the notion . . . of an *effectively calculable* function of positive integers by identifying it with the notion of a recursive function of positive integers."[1] Yiannis Moschovakis ([62]) observes that CT "*refers essentially to the natural numbers*, and so its truth or falsity depends on what they are."

Of course its truth or falsity also depends on what it means for a calculation to be effective, and on whether the definition of recursive function is understood classically or constructively. The latter question leads to *Markov's Principle* MP, which asserts (in its original version) that if an effective algorithm cannot fail to converge then it converges. In its capacity as a principle of unbounded search, MP depends essentially on the natural numbers and implicitly on their order type ω. In classical and constructive mathematical practice, all integers are standard.

Markov's Principle is not generally accepted by constructive mathematicians. Beeson ([2], p. 47) notes that ". . . even [Markov and the Russian constructivists] keep careful track of which theorems depend on it and which are proved without it." Bishop constructivists accept neither CT nor MP, although their work is consistent with both. Kreisel [38] proved that MP is not a theorem of intuitionistic arithmetic. Brouwer and traditional intuitionists reject an analytical version of MP, but their reasoning (cf. Kreisel [43]) has more to do with the nature of the intuitionistic continuum than with what the natural numbers are.

Markov's Rule MR, in contrast, is admissible for most formal systems **T** based on intuitionistic logic.[2] In its simplest form the rule states that if **T** proves that a particular effective algorithm cannot fail to converge, then **T** proves that it converges. Whether or not appropriate versions of the rule are always admissible for constructive theories evidently depends on what a constructive proof is.

Constructive mathematics is often described simply as "mathematics done with intuitionistic logic." While this oversimplification ignores the difference between constructive and classical answers to the question of what constitutes a legitimate mathematical object, it does express a fundamental aspect of constructive proof.

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[1] In 1936 Alan Turing independently proposed that every effectively calculable function can be computed mechanically. It turns out that every general recursive function is Turing computable and vice versa, so CT is also referred to as "the Church-Turing Thesis."

[2] H. Friedman [15] and Dragalin [13] independently discovered a simple method for proving the admissibility of Markov's Rule which applies uniformly to pure and applied intuitionistic systems.

In [21] Arend Heyting quotes Kreisel ([37]) as saying "…the notion of constructive proof is vague." Heyting objects that

> …the notion of vagueness is vague in itself, what we need, is a precise notion of precision. As far as I know, the only notion of this sort is based on a formal system.

But even for a formal system with a recursive proof predicate, "…the difficulty reappears if we ask what it means that a given formula A is provable." Is it enough to derive a contradiction from the assumption that A is unprovable? This question leads back to Markov's Principle, which Heyting did not accept.

More than twenty years after Brouwer rejected the law of excluded middle, Heyting formalized intuitionistic propositional logic and intuitionistic arithmetic. Pure intuitionistic predicate logic was isolated as a proper subsystem of classical predicate logic; cf. Kleene [29] where several equivalent formalizations are studied. Here we follow Kreisel [41] in adopting the acronym **HPC** (for "Heyting predicate calculus") for a formal system of intuitionistic first-order predicate logic.[3] The question of what constitutes a constructive proof *depends essentially on whether* **HPC** *is sound and complete* for its intended intuitionistic interpretation, and this question also involves Markov's Principle, as Gödel and Kreisel observed.

Thus Markov's Principle, Markov's Rule and the common notion of constructive proof are interrelated concepts which matter, at least to constructive mathematicians and logicians, and about which a wide difference of opinion exists. As such they appear to be legitimate candidates for investigation in the spirit of Kreisel's *ideal of informal rigour* (cf. [46]). Our main aim is to clarify all three concepts by presenting and relating a wide variety of examples from the literature. Following Heyting's implicit advice, in this essay we consider only precise versions of MP and MR, and principles weaker than MP, in the context of recursively enumerable formal systems for intuitionistic logic, arithmetic and analysis.

One interesting intermediate principle is known to be equivalent to the *weak* completeness of **HPC** for Beth semantics, as shown in detail by Dyson and Kreisel ([14]).[4] In 1962 Kreisel [41] suggested that this principle "may be provable on the basis of as yet undiscovered axioms which hold for the intended interpretation …*So the problem whether* **HPC** *is weakly complete is still open*."

We show that this "Gödel-Dyson-Kreisel principle" entails the equivalence of the double negations of the binary fan theorem and weak König's Lemma. It follows immediately in a weak common subsystem of intuitionistic and classical analysis from a stronger intermediate principle ("Σ^0_1 double negation shift" for sequences), which refutes weak Church's Thesis and is consistent with Kleene's and Vesley's full system **I** of intuitionistic analysis as presented in [33].[5] Thus *if* Σ^0_1 double negation shift for sequences holds for the intuitionistic theory of numbers and infinitely proceeding sequences, *then* **HPC** is weakly complete for Beth semantics.

[3] Troelstra and van Dalen [75] prefer **IQC**, abbreviating "intuitionistic quantificational calculus," and reserve **IPC** for its propositional subsystem.

[4] A theory is *weakly complete* if and only if it is impossible for an unprovable formula to be valid under the interpretation; cf. Kreisel [36] and §4.2 below.

[5] While "Every sequence is recursive" contradicts Brouwer's fan theorem, weak Church's Thesis (expressing "There can be no nonrecursive sequences") is consistent with **I**; cf. [33], [55].

2. A little history

In the mid twentieth century, first Kleene and then Kreisel developed research programs to axiomatize Brouwer's intuitionistic analysis and establish its consistency. As early as 1941 Kleene recognized a strong affinity between Brouwer's intuitionistic mathematics and classical recursive function theory. In 1950, at the first International Congress of Mathematicians in Cambridge, Massachusetts, Kleene presented his counterexample to Brouwer's fan theorem for the ω-model of analysis in which all infinite sequences of natural numbers are recursive.

At the Summer Institute for Mathematical Logic sponsored by the American Mathematical Society and held at Cornell University, July 1 to August 2, 1957, Kleene spoke on recursive functionals of higher finite types and sketched a new recursive realizability interpretation for a formal system for part of intuitionistic analysis including Brouwer's fan theorem. At the same summer institute Kreisel presented the Kreisel-Lacombe-Shoenfield-Ceitin Theorem (cf. [20]), lectured on continuous functionals, and contrasted Gödel's *Dialectica* interpretation with his own no-counterexample interpretation of Heyting arithmetic (discussed in other chapters in this volume). Later that August, at the International Colloquium "Constructivity in Mathematics" in Amsterdam, Kleene and Kreisel each proposed an interpretation of intuitionistic analysis using "countable" (Kleene, [30]) or continuous (Kreisel [40]) functionals of finite type.

By then Kreisel had proved the independence of the recursive form of Markov's Principle from intuitionistic arithmetic. Markov's Principle for analysis MP_1 says roughly that for *every* infinite sequence α of natural numbers, if it is impossible to prevent α from containing a 0 then $\alpha(n) = 0$ for some natural number n. Kreisel observed in [39] that MP_1 must fail for absolutely free or "lawless" sequences whose values are chosen one by one without restriction, since all properties of lawless sequences are determined by finite initial segments; but Kreisel's theory of lawless sequences was not yet complete, and Brouwer had consciously avoided considering higher order restrictions on choice sequences. The questions whether MP_1 was consistent with, and independent of, intuitionistic analysis remained.

In 1965 Kleene and Vesley [33] provided a formal system **I** for most of Brouwer's theory of natural numbers and arbitrary choice sequences, featuring the axiomatic character of the bar theorem and strengthening Brouwer's assertion that "all full functions are continuous" to an axiom of continuous choice. Using a recursive function-realizability interpretation, Kleene established the consistency of **I** *with or without* MP_1 relative to a classically correct subsystem. A relativized version proved e.g. that **I** is consistent with classical first-order Peano arithmetic (cf. [31]). Inspired by Kreisel's [38], Kleene defined a typed, modified function-realizability interpretation ("special realizability") in order to prove MP_1 independent of **I**.

That same year Kreisel's wide-ranging thoughts on intuitionistic mathematics, emphasizing generalized inductive definitions, "lawlike" sequences, and the possibility of treating choice sequences as a figure of speech, were gathered together in [42]. In 1966 Howard and Kreisel [22] demonstrated the equivalence, over a formal system **H** of "elementary intuitionistic mathematics," of Kleene's formulation of monotone bar induction with a principle of transfinite induction. Their exposition repeated significant parts of the formal development in [33]; details of the overlap are acknowledged in the text and in a footnote provided by Kleene. They also showed that Spector's extension of bar induction to type 1 in [69] could be justified using continuous choice and monotone bar induction at type 0.

A seminal Conference on Intuitionism and Proof Theory was held in August 1968 at the State University of New York in Buffalo. The resulting volume [27] included important contributions by Kreisel (cf. [45]), Bishop, Heyting, Myhill, Dana Scott, Feferman, and logicians of the next generation: Vesley, Dick de Jongh, Martin-Löf, William Howard, Dirk van Dalen, Anne Troelstra and others. Kleene's paper on formalized recursive functionals grew into the technical monograph [32]. Takeuti's course on proof theory and Troelstra's introduction to intuitionism also led to books: [70] and [71], respectively.

Kreisel's axioms for "lawless" sequence (cf. [44]) were improved by Heyting's student Troelstra, who spent most of a year with Kreisel after completing his doctoral dissertation in Amsterdam. They collaborated on [47] (cf. Troelstra's [72]), an extended study of alternative formal systems for intuitionistic analysis emphasizing the roles of lawlike and lawless sequences.

After returning to Amsterdam Troelstra edited, and mostly wrote, a very influential, comprehensive volume [73] on the metamathematics of intuitionistic arithmetic and analysis. Fifteen years later, with van Dalen in Utrecht, he coauthored the indispensable [75]. Troelstra, van Dalen, and Kleene's student de Jongh have taught intuitionism to generations of students from a mostly neutral or classical viewpoint. Wim Veldman, whose own mentor was J. J. de Iongh, still adheres to Brouwer's program of developing mathematics from an intuitionistic perspective.

Most of Kleene's students worked in other areas, although many were influenced by his preference for constructive arguments.[6] Clifford Spector, probably his most brilliant student, died prematurely after extending Brouwer's bar theorem to prove the consistency of classical analysis ([69], edited and prepared for publication by Kreisel). After formalizing his function-realizability interpretations in [32] in order to establish that his formal systems satisfied a precise analogue of Church's Rule,[7] Kleene wrote relatively little on intuitionism.

In 1967 Errett Bishop began his remarkable [6] with a "constructivist manifesto." He acknowledged that Brouwer was correct in his objections to classical logic, but rejected Brouwer's view of the continuum. For Bishop the positive integers were the foundation of mathematics, which must therefore have numerical meaning. He aimed to constructivize mathematical analysis by making every concept affirmative, avoiding abstraction as much as possible, and using only intuitionistic logic, while remaining consistent separately with classical, intuitionistic, and even recursive analysis. His notion of constructive proof was strictly realistic. Like Brouwer, he worked informally and avoided using Markov's Principle, but neoconstructivists working axiomatically may be interested in Markov's Rule.

3. MP and MR in the context of formal systems

In 1958 Kreisel ([38]) developed a modification of Kleene's number-realizability interpretation to show that the primitive recursive form of Markov's Principle, which is (classically) consistent with Heyting arithmetic **HA** by Kleene's number-realizability, is independent of **HA**. Since then logicians have analyzed versions of

[6]Anyone who has studied logic from [29] must have noticed Kleene's use of the superscript $^\circ$ to distinguish theorems whose proofs require classical reasoning from those, such as Gödel's first and second incompleteness theorems, which hold both classically and constructively.

[7]In particular, every closed existential theorem of **I** with or without Markov's Principle can be improved to provide a specific natural number or general recursive sequence as a witness.

MP and MR over intuitionistic predicate calculus and over constructive and intuitionistic theories of natural numbers, real numbers, infinite sequences of natural numbers, and primitive recursive functions of all finite types. While a complete annotated bibliography of these investigations is beyond the scope of this article, in this and following sections we survey the folklore and discuss contributions by many researchers over the past sixty years. Much more information is in the references and their bibliographies.

3.1. **Formal versions of MP.** Over intuitionistic arithmetic **HA** Markov's Principle can be rendered by a formula:

$$\mathrm{MP_0}: \quad \forall e\forall x[\neg\forall y\neg T(e,x,y) \rightarrow \exists yT(e,x,y)]$$

where $\mathrm{T(e,x,y)}$ is a formula expressing Kleene's primitive recursive T-predicate ("y is a gödel number of a successful computation of $\{e\}(x)$"). Alternatively, it can be expressed by a schema:

$$\mathrm{MP_{QF}}: \quad \neg\forall x\neg A(x) \rightarrow \exists xA(x),$$

where $\mathrm{A(x)}$ may contain additional free variables but must be "quantifier-free" (bounded quantifiers are allowed), or

$$\mathrm{MP_{PR}}: \quad \neg\forall n\neg A(n) \rightarrow \exists nA(n)$$

where $\mathrm{A(n)}$ must express a primitive recursive relation of n and its other free variables. Over **HA** all three of $\mathrm{MP_0}$, $\mathrm{MP_{QF}}$ and $\mathrm{MP_{PR}}$ are equivalent.

The most general schematic version of Markov's Principle is

$$\mathrm{MP_D}: \quad \forall x(A(x) \vee \neg A(x)) \ \&\ \neg\forall x\neg A(x) \rightarrow \exists xA(x),$$

(cf. [2], p. 47). Evidently **HA** + $\mathrm{MP_D} \vdash \mathrm{MP_{QF}}$ since if $\mathrm{A(x)}$ has only bounded quantifiers then **HA** $\vdash \forall x(A(x) \vee \neg A(x))$. However, Smorynski ([68], p. 365) proved that **HA** + $\mathrm{MP_{QF}} \nvdash \mathrm{MP_D}$.[8] *A fortiori* $\mathrm{MP_D}$ is not provable in **HPC** from instances of $(\neg\forall x\neg P(x) \rightarrow \exists xP(x))$ with $\mathrm{P(x)}$ prime. We leave the proof that **HPC** + $\mathrm{MP_D} \nvdash (\neg\forall x\neg P(x) \rightarrow \exists xP(x))$ as an exercise for the reader.

As principles of pure predicate logic, $\mathrm{MP_D}$ and variants can have no convincing constructive justification because the connection with the natural numbers is lost. Closed instances are *persistently consistent* with **HPC** in the sense that their double negations are provable, but the schemas are not persistently consistent because **HPC** $\nvdash \neg\neg\forall y[\forall x(P(x,y) \vee \neg P(x,y)) \ \&\ \neg\forall x\neg P(x,y) \rightarrow \exists xP(x,y)]$.[9]

Over a two-sorted theory such as Kleene's **I** ([33]) and subsystems, Troelstra's **EL** ([73]) or Veldman's **BIM** ([79]), Markov's Principle can be strengthened to

$$\mathrm{MP_1}: \quad \forall\alpha(\neg\forall x\neg\alpha(x) = 0 \rightarrow \exists x\alpha(x) = 0).$$

As long as the characteristic function of Kleene's T predicate is adequately represented in the system, $\mathrm{MP_1}$ entails $\mathrm{MP_0}$, and if the axiom schema of countable (unique) choice for quantifier-free relations is present then $\mathrm{MP_1}$ entails $\mathrm{MP_{QF}}$. Whether or not $\mathrm{MP_1}$ entails $\mathrm{MP_D}$ depends on whether or not the system proves that every decidable relation has a characteristic function; cf. [77], [76], [61]. The type-0 variable x is always intended to range over the natural numbers, but the

[8] In the presence of Church's Thesis, $\mathrm{MP_{QF}}$ and $\mathrm{MP_D}$ are arithmetically equivalent. If CT! is the recursive comprehension principle $\forall x\exists!yA(x,y) \rightarrow \exists e\forall x\exists y[T(e,x,y) \ \&\ A(x,U(y))]$ (where $\exists!yB(y)$ abbreviates $\exists yB(y) \ \&\ \forall z(B(y) \ \&\ B(z) \rightarrow y = z)$), then **HA** + CT! + $\mathrm{MP_{QF}} \vdash \mathrm{MP_D}$.

[9] For a counterexample, consider a linear Kripke model with root 0, nodes $n \in \omega$ with $n < n+1$, $D(n) = \{\mathbf{0}, \dots, \mathbf{n}\}$ and $\mathrm{P}(\mathbf{n+1}, \mathbf{n})$ true at all $k \geq n+1$.

type-1 variable α may be interpreted as ranging over all infinite sequences of natural numbers or over a proper subset (or subspecies) of them; when α ranges over the recursive sequences, MP_1 expresses Markov's original intention.

3.2. **Markov's Rule and constructive arithmetical truth.** By the (uniform, syntactic) Friedman-Dragalin translation, every formal system mentioned so far has the property that if $\forall x(A(x) \vee \neg A(x))$ & $\neg\forall x \neg A(x)$ is provable so is $\exists x A(x)$; so Markov's Rule $\mathrm{MR_D}$ is admissible for these (and most other) formal systems based on intuitionistic logic. In [48] Leivant proved that $\mathrm{MR_D}$ holds for a system $\mathbf{HA}^*$ ($\mathbf{HA}$ extended with transfinite induction over all recursive well-orderings) which he suggests may capture the notion of constructive arithmetical truth.

Any attempt to identify constructive arithmetical truth with provability in a consistent recursively enumerable extension of $\mathbf{HA}$ will of course be frustrated by Gödel's incompleteness theorem. However, over $\mathbf{HA}$ or $\mathbf{I}$ the conclusion of $\mathrm{MR_{QF}}$ for a closed formula $\exists x A(x)$ can be strengthened to "$A(\mathbf{n})$ is provable for some numeral $\mathbf{n}$" because these systems satisfy *numerical existential instantiation*: If a closed formula $\exists x E(x)$ is provable in the system, then $E(\mathbf{n})$ is provable for some numeral $\mathbf{n}$ ([28], [65], [32]). In particular, the (standard) natural numbers suffice as witnesses for true existential statements of intuitionistic arithmetic.

Adding $\mathrm{MP_{QF}}$ to intuitionistic arithmetic or analysis preserves numerical existential instantiation for Σ^0_1 formulas and does not increase the stock of provably recursive functions. Peano arithmetic is Π^0_2-conservative over $\mathbf{HA}$ by the Gödel-Gentzen negative translation with $\mathrm{MR_{QF}}$, and $\mathbf{I} + \mathrm{MP_{QF}}$ is Π^0_2-conservative over $\mathbf{B}$ and $\mathbf{I}$ by [60]. These facts argue for the constructive truth of the arithmetical forms of Markov's Principle.

3.3. **A note on metamathematical methods.** In his case study on informal rigour [46], Kreisel considers the possibility that there may be "simple conditions, easily verified for current intuitionistic systems, that imply easily the consistency of CT and closure under Church's rule." If so, the need for "detailed studies like ingenious realizability interpretations" would be eliminated.

The Friedman-Dragalin translation is a simple tool to prove intuitionistic formal systems are closed under Markov's rule. Coquand and Hoffmann [11] showed how the method can be extended to prove that $\mathrm{MP_{QF}}$ is Π^0_2-conservative over many intuitionistic systems. Kripke models of $\mathbf{HPC}$ and $\mathbf{HA}$, and topological and Beth models of theories of choice sequences, help to establish the relative consistency and independence of MP and related principles.

But for delicate questions about the strength of variants and weakenings of Markov's principle over subsystems of intuitionistic analysis, realizability interpretations (Kleene and Nelson's number-realizability, Kreisel's modified number-realizability, Kleene's function-realizability, ${}_{\mathcal{S}}$realizability [33] and realizability-plus-truth, the author's ^{G}realizability [55], Lifschitz realizability . . .) are among the best tools available. Some of these methods can be replaced by categorical ones (cf. [78]), which we do not use here.

4. Markov's principle and completeness properties of HPC

4.1. **Markov's principle and (strong) completeness of HPC.** An analytical consequence of $\mathrm{MP_{PR}}$ is intuitionistically equivalent to the completeness of $\mathbf{HPC}$ for the topological and Beth interpretations of intuitionistic logic. This fact is due to Gödel and Kreisel ([36]). In order to state it precisely, let β range over infinite (free choice) sequences of natural numbers and let $\mathrm{B}(\beta)$ abbreviate $\forall \mathrm{n}(\beta(\mathrm{n}) \leq 1)$. Then the following statements are equivalent:[10]

(1) $\forall \beta_{\mathrm{B}(\beta)} \neg \forall \mathrm{n} \neg \mathrm{A}(\mathrm{n}, \beta) \rightarrow \forall \beta_{\mathrm{B}(\beta)} \exists \mathrm{n} \mathrm{A}(\mathrm{n}, \beta)$ holds for all primitive recursive relations $\mathrm{A}(\mathrm{n}, \beta)$ between natural numbers and free choice sequences.

(2) **HPC** is complete for its topological interpretation and for Beth's semantic construction of intuitionistic logic [5]. Every formula of the predicate language which is valid under either interpretation is provable in **HPC**.

Evidently $\mathrm{MP}_{\mathrm{PR}}$ (with a choice sequence variable β free) entails (1), and (1) (with a choice sequence variable α free) entails MP_1 by the following argument. Let $\mathrm{A}(\alpha, \mathrm{n}, \beta)$ abbreviate $\exists \mathrm{m} < \overline{\beta}(\mathrm{n})(\alpha(\mathrm{m}) = 0)$, where $\overline{\beta}(\mathrm{n})$ abbreviates $\Pi_{\mathrm{m}<\mathrm{n}} \mathrm{p}_{\mathrm{m}}^{\beta(\mathrm{m})+1}$ and p_{m} represents the mth prime number, with $\mathrm{p}_0 = 2$. Assume (a) $\neg \forall \mathrm{x} \neg \alpha(\mathrm{x}) = 0$. Then also (b) $\forall \beta_{\mathrm{B}(\beta)} \neg \forall \mathrm{n} \neg \mathrm{A}(\alpha, \mathrm{n}, \beta)$ so $\forall \beta_{\mathrm{B}(\beta)} \exists \mathrm{n} \mathrm{A}(\alpha, \mathrm{n}, \beta)$ by (1), and therefore (c) $\exists \mathrm{x} \alpha(\mathrm{x}) = 0$.

4.2. The Gödel-Dyson-Kreisel principle and weak completeness of HPC.

Verena Huber-Dyson and Kreisel verified in [14] that the *double negation* of (1) entails the *weak* completeness of **HPC** (in the sense that unprovable formulas cannot be valid) for Beth's semantics, and Kreisel [41] proved the converse. The name "Gödel-Dyson-Kreisel Principle" seems appropriate for the schema

$$\forall \beta_{\mathrm{B}(\beta)} \neg \forall \mathrm{n} \neg \mathrm{A}(\mathrm{n}, \beta) \rightarrow \neg\neg \forall \beta_{\mathrm{B}(\beta)} \exists \mathrm{n} \mathrm{A}(\mathrm{n}, \beta)$$

where $\mathrm{A}(\mathrm{n}, \beta)$ expresses a primitive recursive relation of n, β. Primitive recursive relations depend only on finite initial segments of their choice sequence variables, so the principle can be expressed by the schema

$$\mathrm{GDK}_{\mathrm{PR}}: \quad \forall \beta_{\mathrm{B}(\beta)} \neg\neg \exists \mathrm{n}\ \mathrm{u}(\overline{\beta}(\mathrm{n})) = 0 \rightarrow \neg\neg \forall \beta_{\mathrm{B}(\beta)} \exists \mathrm{n}\ \mathrm{u}(\overline{\beta}(\mathrm{n})) = 0$$

where the u may represent any primitive recursive function.

From the point of view on intuitionistic analysis taken by Kleene and Vesley in [33] it seems unnecessary to restrict the function represented by u to be primitive recursive, so consider also the principle

$$\mathrm{GDK}: \quad \forall \beta_{\mathrm{B}(\beta)} \neg\neg \exists \mathrm{n}\, \rho(\overline{\beta}(\mathrm{n})) = 0 \rightarrow \neg\neg \forall \beta_{\mathrm{B}(\beta)} \exists \mathrm{n}\, \rho(\overline{\beta}(\mathrm{n})) = 0.$$

Theorem 1. *Assuming the double negation* $\neg\neg\mathrm{FT}_1$ *of the following form of the classically and intuitionistically correct binary fan theorem*[11]

$$\mathrm{FT}_1: \quad \forall \beta_{\mathrm{B}(\beta)} \exists \mathrm{x}\, \rho(\overline{\beta}(\mathrm{x})) = 0 \rightarrow \exists \mathrm{n} \forall \beta_{\mathrm{B}(\beta)} \exists \mathrm{x} \leq \mathrm{n}\, \rho(\overline{\beta}(\mathrm{x})) = 0,$$

the double negation $\neg\neg\mathrm{WKL}$ *of weak König's Lemma*

$$\mathrm{WKL}: \quad \forall \mathrm{n} \exists \beta_{\mathrm{B}(\beta)} \forall \mathrm{x} \leq \mathrm{n}\, \rho(\overline{\beta}(\mathrm{x})) \neq 0 \rightarrow \exists \beta_{\mathrm{B}(\beta)} \forall \mathrm{x} \rho(\overline{\beta}(\mathrm{x})) \neq 0$$

is equivalent to GDK *over two-sorted intuitionistic arithmetic.*

Proof. We can actually prove $\neg\neg\mathrm{FT}_1(\rho)\ \&\ \mathrm{GDK}(\rho) \leftrightarrow \neg\neg\mathrm{WKL}(\rho)$ in two-sorted intuitionistic arithmetic $\mathbf{IA}_1$, a weak subsystem of intuitionistic recursive analysis obtained from Kleene's **I** by omitting the axiom schemas of countable choice, bar

[10] The proof requires the fan theorem, defined in the next subsection, but does not use classically false continuity principles.

[11] While FT_1 is false for recursive sequences β, it is true under the intended intuitionistic interpretation when β ranges over all binary choice sequences. By "the double negation of the fan theorem" we mean $\neg\neg\mathrm{FT}_1(\rho)$ rather than $\neg\neg\forall\rho\mathrm{FT}_1(\rho)$, and similarly for $\neg\neg\mathrm{WKL}$.

induction and continuous choice.[12] (By a similar proof, WKL is equivalent over $\mathbf{IA}_1$ to the conjunction of FT_1 and MP_1.)

First consider $\neg\neg\mathrm{FT}_1(\rho)$ & $\mathrm{GDK}(\rho) \to \neg\neg\mathrm{WKL}(\rho)$. Since $\neg\neg\mathrm{WKL}(\rho)$ is equivalent in $\mathbf{IA}_1$ to $\neg\neg\forall \mathrm{n}\exists\beta_{\mathrm{B}(\beta)}\forall \mathrm{x} \leq \mathrm{n}\,\rho(\overline{\beta}(\mathrm{x})) \neq 0 \to \neg\neg\exists\beta_{\mathrm{B}(\beta)}\forall \mathrm{x}\rho(\overline{\beta}(\mathrm{x})) \neq 0$ it will be enough to prove $\neg\exists\beta_{\mathrm{B}(\beta)}\forall \mathrm{x}\rho(\overline{\beta}(\mathrm{x})) \neq 0 \to \neg\forall \mathrm{n}\exists\beta_{\mathrm{B}(\beta)}\forall \mathrm{x} \leq \mathrm{n}\,\rho(\overline{\beta}(\mathrm{x})) \neq 0$.

Assume (a) $\neg\exists\beta_{\mathrm{B}(\beta)}\forall \mathrm{x}\,\rho(\overline{\beta}(\mathrm{x})) \neq 0$. Then (b) $\forall\beta_{\mathrm{B}(\beta)}\neg\neg\exists \mathrm{x}\,\rho(\overline{\beta}(\mathrm{x})) = 0$, so by $\mathrm{GDK}(\rho)$: (c) $\neg\neg\forall\beta_{\mathrm{B}(\beta)}\exists \mathrm{x}\,\rho(\overline{\beta}(\mathrm{x})) = 0$. Then (d) $\neg\neg\exists \mathrm{n}\forall\beta_{\mathrm{B}(\beta)}\exists \mathrm{x} \leq \mathrm{n}\,\rho(\overline{\beta}(\mathrm{x})) = 0$ by $\neg\neg\mathrm{FT}_1(\rho)$, so (e) $\neg\forall \mathrm{n}\neg\forall\beta_{\mathrm{B}(\beta)}\exists \mathrm{x} \leq \mathrm{n}\,\rho(\overline{\beta}(\mathrm{x})) = 0$. Because there are only finitely many binary sequences of length $\leq n$, $\mathbf{IA}_1$ proves that $\neg\forall\beta_{\mathrm{B}(\beta)}\exists \mathrm{x} \leq \mathrm{n}\,\rho(\overline{\beta}(\mathrm{x})) = 0$ is equivalent to $\exists\beta_{\mathrm{B}(\beta)}\forall \mathrm{x} \leq \mathrm{n}\,\rho(\overline{\beta}(\mathrm{x})) \neq 0$, which together with (e) gives the desired conclusion.

For the converse, assume (f) $\forall\beta_{\mathrm{B}(\beta)}\neg\neg\exists \mathrm{x}\,\rho(\overline{\beta}(\mathrm{x})) = 0$, so (equivalently in $\mathbf{IA}_1$) (g) $\neg\exists\beta_{\mathrm{B}(\beta)}\forall \mathrm{x}\,\rho(\overline{\beta}(\mathrm{x})) \neq 0$. Then (h) $\neg\forall \mathrm{n}\exists\beta_{\mathrm{B}(\beta)}\forall \mathrm{x} \leq \mathrm{n}\rho(\overline{\beta}(\mathrm{x})) \neq 0$ by $\neg\neg\mathrm{WKL}(\rho)$, and this is equivalent in $\mathbf{IA}_1$ to (i) $\neg\forall \mathrm{n}\neg\forall\beta_{\mathrm{B}(\beta)}\exists \mathrm{x} \leq \mathrm{n}\,\rho(\overline{\beta}(\mathrm{x})) = 0$ and hence to (j) $\neg\neg\exists \mathrm{n}\forall\beta_{\mathrm{B}(\beta)}\exists \mathrm{x} \leq \mathrm{n}\,\rho(\overline{\beta}(\mathrm{x})) = 0$. Then *a fortiori* (k) $\neg\neg\forall\beta_{\mathrm{B}(\beta)}\exists \mathrm{x}\,\rho(\overline{\beta}(\mathrm{x})) = 0$, completing the proof of $\neg\neg\mathrm{WKL}(\rho) \to \mathrm{GDK}(\rho)$ in $\mathbf{IA}_1$. The proof in $\mathbf{IA}_1$ of $\neg\neg\mathrm{WKL}(\rho) \to \neg\neg\mathrm{FT}_1(\rho)$ is similar. □

This equivalence is implicit in [14]. As a consequence, if **HPC** is weakly complete for its intended interpretation then weak König's Lemma is *persistently consistent* with the fan theorem in the sense that any counterexample to either would generate a counterexample to the other.[13] Kleene's counterexample for recursive sequences (Lemma 9.8 of [33]) applies equally to both.

Finally, it is worth mentioning that GDK is equivalent over $\mathbf{F} = \mathbf{IA}_1 + \mathrm{FT}_1$ to the Gödel-Gentzen negative interpretation of FT_1, so $\mathbf{F}$ + GDK is the minimum classical extension of $\mathbf{F}$.

5. Consistency and independence results

Because it is provable in $\mathbf{I} + \mathrm{MP}_1$, GDK is evidently consistent with $\mathbf{I}$. In fact $\mathbf{I}$ + GDK lies strictly between $\mathbf{I}$ and $\mathbf{I} + \mathrm{MP}_1$, but we can say more. Consider Vesley's Schema:

$$\mathrm{VS}: \quad \forall\alpha\forall \mathrm{x}\exists\beta(\overline{\beta}(\mathrm{x}) = \overline{\alpha}(\mathrm{x}) \ \&\ \neg\mathrm{A}(\beta)) \ \&\ \forall\alpha(\neg\mathrm{A}(\alpha) \to \exists\beta\mathrm{B}(\alpha,\beta)) \\ \to \forall\alpha\exists\beta(\neg\mathrm{A}(\alpha) \to \mathrm{B}(\alpha,\beta)).$$

In [80] Vesley proposed VS as a "palatable substitute" for Kripke's Schema KS, which proves Brouwer's "creating subject" counterexamples but conflicts with the $\forall\alpha\exists\beta$-continuity axiom schema of $\mathbf{I}$.[14] Vesley proved that $\mathbf{I}$ + VS is consistent (by special-realizability) and refutes a large number of classical principles objected to by Brouwer, including Markov's Principle.

Because it is special-realizable, GDK is consistent with $\mathbf{I}$ + VS. Assuming **HPC** is weakly complete for Beth semantics, GDK is a palatable substitute for Markov's Principle MP_1 and the formal system $\mathbf{I}$ + VS + GDK has real advantages over $\mathbf{I}$ as an axiomatization of intuitionistic mathematical practice.

[12] See [61] for a precise definition of $\mathbf{IA}_1$. As it only proves the existence of primitive recursive functions, $\mathbf{IA}_1$ is weaker than Troelstra's **EL**.

[13] The useful compound adjective "persistently consistent" may be due to Artemov.

[14] KS is consistent with $\forall\alpha\exists \mathrm{x}$-continuity and with $\forall\alpha\exists!\beta$-continuity, but KS and MP_1 together entail the law of excluded middle; cf. [63, 64], [56].

Theorem 2. GDK *is consistent with* **I** *and with Vesley's Schema* VS*, which proves Brouwer's "creating subject" counterexamples. Hence* GDK *does not entail* MP_1*.*

Proof. GDK is both Kleene function-realizable and ${}_S$realizable. Vesley's Schema is ${}_S$realizable so **I** + VS + GDK is consistent. MP_1 is realizable but not ${}_S$realizable, so is not provable in **I** + VS + GDK. (In fact, **I** + VS + MP_1 is inconsistent.) □

Theorem 3. GDK *is independent of* $\mathbf{I}+\mathrm{VS}$.

Proof. Every theorem of **I** + VS is ^{G}realizable in the sense of the author's [55], but GDK is not. □

By ^{G}realizability, $\mathbf{I}+\mathrm{VS}$ is consistent with "weak Church's thesis" $\forall\alpha\neg\neg\mathrm{GR}(\alpha)$, where $\mathrm{GR}(\alpha)$ is $\exists \mathrm{e}\forall \mathrm{x}\exists \mathrm{y}(\mathrm{T}(\mathrm{e},\mathrm{x},\mathrm{y})\ \&\ \mathrm{U}(\mathrm{y})=\alpha(\mathrm{x}))$. Thus GDK is not provable in $\mathbf{I}+\mathrm{VS}+\forall\alpha\neg\neg\mathrm{GR}(\alpha)$.

6. Markov's principle and order in the continuum

6.1. **MP and the intuitionistic continuum.** Brouwer's theory of the continuum is closely related to his treatment of Baire and Cantor space, and MP_1 can be expressed in terms of real numbers. Relying on sources by Brouwer and Heyting, in Chapter 3 of [33] Vesley gave a precise formal development of the nonnegative intuitionistic reals as equivalence classes, under coincidence, of *real number generators*. An r.n.g. is a Cauchy sequence $\alpha(0), \alpha(1)/2^1, \alpha(2)/2^2, \alpha(3)/2^3, \ldots$ of dual fractions, determined completely by the sequence α of numerators satisfying the Cauchy condition $\forall k\exists x\forall p 2^k|2^p\alpha(x)-\alpha(x+p)|<2^{x+p}$.

A *canonical real number generator* (c.r.n.g.) is an r.n.g. satisfying the uniform Cauchy condition $\forall x|2\alpha(x)-\alpha(x+1)|\leq 1$, which is much easier to work with formally. The fact that every r.n.g. coincides with a c.r.n.g. is provable using countable choice, but Vesley formalized a significant part of intuitionistic real analysis in terms of r.n.g. We follow his treatment, based on Brouwer [8], for the definitions and a few basic properties of four intuitionistic relations between real number generators: coincidence $\overset{\circ}{=}$, apartness $\#$, virtual ordering $\dot{<}$ and the measurable natural ordering $<\!\circ$.[15]

Each of the following formulas defines the relation on the left side of the $\leftrightarrow$.

- $\alpha\in\mathbb{R}\leftrightarrow\forall \mathrm{k}\exists \mathrm{x}\forall \mathrm{p}2^{\mathrm{k}}|2^{\mathrm{p}}\alpha(\mathrm{x})-\alpha(\mathrm{x}+\mathrm{p})|<2^{\mathrm{x}+\mathrm{p}}$
- $\alpha\overset{\circ}{=}\beta\leftrightarrow\forall \mathrm{k}\exists \mathrm{x}\forall \mathrm{p}2^{\mathrm{k}}|\alpha(\mathrm{x}+\mathrm{p})-\beta(\mathrm{x}+\mathrm{p})|<2^{\mathrm{x}+\mathrm{p}}$.
- $\alpha\,\#\,\beta\leftrightarrow\exists \mathrm{k}\exists \mathrm{x}\forall \mathrm{p}2^{\mathrm{k}}|\alpha(\mathrm{x}+\mathrm{p})-\beta(\mathrm{x}+\mathrm{p})|>2^{\mathrm{x}+\mathrm{p}}$.
- $\alpha<\!\circ\,\beta\leftrightarrow\exists \mathrm{k}\exists \mathrm{x}\forall \mathrm{p}2^{\mathrm{k}}(\beta(\mathrm{x}+\mathrm{p})\dot{-}\alpha(\mathrm{x}+\mathrm{p}))\geq 2^{\mathrm{x}+\mathrm{p}}$.
- $\alpha\,\dot{<}\,\beta\leftrightarrow\neg\beta<\!\circ\,\alpha\ \&\ \neg\alpha\overset{\circ}{=}\beta$.
- $\alpha\in\mathbb{R}'\leftrightarrow\forall \mathrm{x}|2\alpha(\mathrm{x})-\alpha(\mathrm{x}+1)|\leq 1$.

Vesley derived hundreds of equivalences in $\mathbf{IA}_1$ plus countable choice, including

(1) $\forall\alpha_{\alpha\in\mathbb{R}}\forall\beta_{\beta\in\mathbb{R}}(\alpha\,\dot{<}\,\beta\leftrightarrow\neg\neg\alpha<\!\circ\,\beta)$.

(2) $\forall\alpha_{\alpha\in\mathbb{R}}\forall\beta_{\beta\in\mathbb{R}}(\alpha\overset{\circ}{=}\beta\leftrightarrow\neg\alpha\,\dot{<}\,\beta\ \&\ \neg\alpha\,\dot{>}\,\beta)$.

[15] Heyting (cf. [19], page 107) called the measurable natural ordering the "pseudo-ordering" and symbolized it by "$<$" instead of Brouwer's "$<\!\circ$." Bishop retained Heyting's symbol for the measurable natural order, which agrees with its classical use, but Mandelkern [52] introduced the term "pseudo-positive" for a (constructively) weaker relation he symbolized by "$0<\!\circ$," and changed Heyting's "$\dot{<}$" for the virtual ordering to "$<\cdot$." The resulting notational disconnect with the careful treatment in [33] has probably resulted in much duplication of Vesley's and Kleene's work by modern constructivists.

(3) $\forall\alpha_{\alpha\in\mathbb{R}}\forall\beta_{\beta\in\mathbb{R}}(\alpha \dot{<} \beta \leftrightarrow \neg\alpha \dot{>} \beta \ \&\ \neg\alpha \overset{\circ}{=} \beta)$.
(4) $\forall\alpha_{\alpha\in\mathbb{R}}\exists\beta_{\beta\in\mathbb{R}'}(\alpha \overset{\circ}{=} \beta)$.

Kleene ([33] Chapter 4) analyzed the logical relationships among various order properties of real numbers considered by Brouwer in [9].[16] He proved in particular that MP_1 is interderivable over $\mathbf{IA}_1$ with each of the following statements:[17]

(a) $\forall \mathrm{z}\forall\alpha[\neg\forall \mathrm{y}\neg \mathrm{T}_0^1(\widetilde{\alpha}\,(\mathrm{y}), \mathrm{z}, \mathrm{y}) \to \exists \mathrm{y}\mathrm{T}_0^1(\widetilde{\alpha}\,(\mathrm{y}), \mathrm{z}, \mathrm{y})]$
(b) $\forall\alpha_{\alpha\in\mathbb{R}}\forall\beta_{\beta\in\mathbb{R}}(\neg\alpha \overset{\circ}{=} \beta \leftrightarrow \alpha \# \beta)$
(c) $\forall\alpha_{\alpha\in\mathbb{R}}\forall\beta_{\beta\in\mathbb{R}}(\alpha \dot{<} \beta \leftrightarrow \alpha <\!\circ\, \beta)$

Kleene also showed that $\forall\alpha_{\alpha\in\mathbb{R}}\forall\beta_{\beta\in\mathbb{R}}(\neg\alpha \overset{\circ}{=} \beta \to \alpha \# \beta)$ is equivalent over $\mathbf{IA}_1$ to $\forall\alpha_{\alpha\in\mathbb{R}}\forall\beta_{\beta\in\mathbb{R}}(\alpha \dot{<} \beta \to \alpha <\!\circ\, \beta)$, and that neither is derivable in $\mathbf{I}$.

6.2. **MP and the constructive continuum.** Working informally and avoiding the use of negation, Bishop [6] represented constructive real numbers by *regular* Cauchy sequences $\{x_n\}$ of rational numbers satisfying $|x_n - x_m| < 1/n + 1/m$. He did not accept Markov's principle so the distinctions described above are meaningful for his constructive reals. Let $\mathbb{R}''$ be the class of regular Cauchy sequences $x = \{x_n\}$. It is straightforward to prove (using countable choice, which Bishop accepted) that every Cauchy sequence coincides with a regular one, so $\mathbb{R}$, $\mathbb{R}'$ and $\mathbb{R}''$ represent the same constructive reals.[18]

Mandelkern [52] studied order properties of the constructive reals, deriving equivalences between them and properties of the *decision sequences* (monotone nondecreasing binary sequences) involved in Brouwer's creating subject counterexamples. He formulated a "limited continuity principle" LCP for functions and an equivalent "almost separating principle" ASP for real numbers, which follows from the "limited principle of existence" LPE, a version of Markov's principle.[19]

In [53] Mandelkern showed that LPE is equivalent to the conjunction of the "weak limited principle of existence" WLPE (which is ASP) and the "lesser limited principle of existence" LLPE. He stated these principles using "$<$" instead of "$<\!\circ$" and "$<\cdot$" instead of "$\dot{<}$", but in Vesley's notation LPE (equivalent to MP_1 by [33]) is $\forall\alpha_{\alpha\in\mathbb{R}''}(0 \dot{<} \alpha \to 0 <\!\circ\, \alpha)$; WLPE is $\forall\beta_{\beta\in\mathbb{R}''}(\forall\alpha_{\alpha\in\mathbb{R}''}(0 \dot{<} \alpha \vee \alpha \dot{<} \beta) \to 0 <\!\circ\, \beta)$; and LLPE is $\forall\beta_{\beta\in\mathbb{R}''}(\neg\neg 0 <\!\circ\, \beta \to \forall\alpha_{\alpha\in\mathbb{R}''}(0 \dot{<} \alpha \vee \alpha \dot{<} \beta))$.

Then in [24] Ishihara decomposed MP_1 into "weak Markov's principle"

$$\mathrm{WMP}: \ \forall\alpha[\forall\beta(\neg\neg\exists \mathrm{n}(\beta(\mathrm{n}) \neq 0) \vee \neg\neg\exists \mathrm{n}(\alpha(\mathrm{n}) \neq 0 \ \&\ \beta(\mathrm{n}) = 0)) \to \exists \mathrm{n}(\alpha(\mathrm{n}) \neq 0)]$$

and "disjunctive Markov's principle"

$$\mathrm{MP}^{\vee}: \ \forall\alpha\forall\beta[\neg\neg\exists \mathrm{n}(\alpha(\mathrm{n}) = 0 \vee \beta(\mathrm{n}) = 0) \to \neg\neg\exists \mathrm{n}\alpha(\mathrm{n}) = 0 \vee \neg\neg\exists \mathrm{n}\beta(\mathrm{n}) \neq 0]$$

equivalent respectively to Mandelkern's WLPE and LLPE. Their main interest was to discover which theorems of classical real analysis could (only) be proved in their original form by assuming these and other principles considered to be nonconstructive. Much more has been done along these lines (cf. [7] and [25]).

[16] The diagram on page 177 gives five equivalents of $\forall\alpha_{\alpha\in\mathbb{R}}\forall\beta_{\beta\in\mathbb{R}}(\neg\alpha \overset{\circ}{=} \beta \to \alpha \# \beta)$ over $\mathbf{I}$.

[17] (a) extends M_0 from recursive functions to recursive functionals, while (b) and (c) confirm that acceptance of MP_1 would greatly simplify the intuitionistic theory of the continuum.

[18] Vesley's treatment can easily be extended informally to all intuitionistic real numbers, so 0 loses its special status as a boundary point.

[19] The limited continuity principle says that every real-valued nondecreasing function on the closed unit interval which approximates intermediate values is continuous.

7. The descriptive power of MP and MR in the context of analysis

Many studies of the consequences and relative independence of variants and weakenings of Markov's principle in the context of systems with intuitionistic logic have appeared since Kreisel's [38]. Among these, in rough chronological order, are Myhill [63] and [64]; Vesley [81]; Troelstra [74]; Luckhardt [50] and [51]; Beeson [3]; Mandelkern [52] and [53]; Bridges and Richman [7]; Ishihara [24] and [25]; Scedrov and Vesley [66]; Coquand and Hoffman [11]; Ishihara and Mines [26]; Kohlenbach [34] and [35]; Akama, Berardi, Hayashi and Kohlenbach [1]; Moschovakis [58]; Loeb [49]; Herbelin [18]; Ilik and Nakata [23]; Fujiwara, Ishihara and Nemoto [16]; Hendtlass and Lubarsky [17]; Coquand and Mannaa [12].

Even when obtained by classical methods, technical results of this kind can be interpreted in terms of the models permitted or eliminated by versions of MP, as already observed for $\mathrm{MP}_{\mathrm{QF}}$ in the context of arithmetic and GDK in the context of predicate logic. This section considers some consequences of adding MP_1, or a weak version of MP_1 such as $\mathrm{MP}^{\vee}$ or WMP or GDK, to two-sorted intuitionistic systems between $\mathbf{IA}_1$ and $\mathbf{I}$ which already obey Markov's rule.

7.1. Constructive existence and the Church-Kleene Rule.

$\forall\alpha\mathrm{GR}(\alpha)$ is false in $\mathbf{I}$ by $\forall\alpha\exists x$-continuous choice, and is inconsistent with the classically correct subsystem $\mathbf{F} = \mathbf{IA}_1 + \mathrm{FT}_1$ of $\mathbf{I}$ by Lemma 9.8 of [33]. Even without continuity principles Brouwer's choice sequences cannot all be recursive.

Nevertheless, as Kleene proved in [32], if $\exists\alpha\mathrm{A}(\alpha)$ is closed and $\mathbf{I} \vdash \exists\alpha\mathrm{A}(\alpha)$ then $\mathbf{I} \vdash \exists\alpha[\mathrm{GR}(\alpha)\ \&\ \mathrm{A}(\alpha)]$.[20] This *Church-Kleene Rule* extends to $\mathbf{I} + \mathrm{MP}_1$; so every infinite sequence $\mathbf{I}$ or $\mathbf{I} + \mathrm{MP}_1$ can prove exists must be recursive.

The Church-Kleene Rule holds also for $\mathbf{B}$ and $\mathbf{B} + \mathrm{MP}_1$ where $\mathbf{B}$ is the classically correct subsystem of $\mathbf{I}$ which omits continuous choice. All these systems satisfy numerical instantiation. By [32] with [60], every closed theorem of the form $\exists\alpha\mathrm{A}(\alpha)$ can be improved to $\mathrm{A}(\{\mathbf{e}\})$ for some numeral $\mathbf{e}$ for which $\mathbf{B}$ proves that e is the gödel number of a total recursive function. In this sense the natural numbers suffice to determine witnesses for existential assertions, not only in intuitionistic arithmetic but also in constructive and intuitionistic analysis.

7.2. Unavoidability of arithmetical sequences.

The situation is different for *unavoidable existence*.[21] On the one hand, ^{G}realizability (cf. [55]) establishes that $\forall\alpha\neg\neg\mathrm{GR}(\alpha)$ (hence $\neg\exists\alpha\neg\mathrm{GR}(\alpha)$) is consistent with $\mathbf{I}$ extended by

$$\Sigma^0_1\text{-}\mathrm{DNS}_0: \quad \forall\alpha[\forall \mathrm{x}\neg\neg\exists \mathrm{y}\alpha(\langle \mathrm{x},\mathrm{y}\rangle) = 0 \rightarrow \neg\neg\forall \mathrm{x}\exists \mathrm{y}\alpha(\langle \mathrm{x},\mathrm{y}\rangle) = 0],$$

a version of the consequence of MP studied in [66].

On the other hand, Solovay showed that $\mathbf{S} + \mathrm{MP}_1$ proves that characteristic functions for all arithmetical relations cannot fail to exist (so in particular $\mathbf{S} + \mathrm{MP}_1 \vdash \neg\neg\exists\alpha\neg\mathrm{GR}(\alpha)$), where $\mathbf{S}$ is a classically correct subsystem of $\mathbf{I}$ including the bar induction schema BI_1 and countable choice for arithmetical relations.[22]

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[20] In fact, under the same assumption there is a numeral $\mathbf{e}$ such that $\mathbf{I}$ proves a sentence which could be abbreviated by $\forall \mathrm{x}\{\mathbf{e}\}(\mathrm{x})\downarrow\ \&\ \mathrm{A}(\{\mathbf{e}\})$, cf. page 101 of [32].

[21] "Unavoidable existence" is represented formally by the doubly negated existential quantifier. If $\neg\neg\exists \mathrm{x}\mathrm{A}(\mathrm{x})$ is a closed theorem of e.g. $\mathbf{HA}$, then no consistent extension of $\mathbf{HA}$ can prove $\neg\exists \mathrm{x}\mathrm{A}(\mathrm{x})$ so $\exists \mathrm{x}\mathrm{A}(\mathrm{x})$ is persistently consistent with $\mathbf{HA}$.

[22] His proof is described in [57], and extended to show that all classically Δ^1_1 sequences are unavoidable in $\mathbf{S} + \mathrm{MP}_1$.

Part of Solovay's argument goes through when MP_1 is replaced by the principle

$$\Sigma^0_1\text{-DNS}_1 : \quad \forall\rho[\forall\alpha\neg\neg\exists \mathrm{x}\rho(\overline{\alpha}(\mathrm{x})) = 0 \rightarrow \neg\neg\forall\alpha\exists \mathrm{x}\rho(\overline{\alpha}(\mathrm{x})) = 0]$$

which entails GDK but not MP_1. In $\mathbf{S} + \Sigma^0_1\text{-DNS}_1$ and *a fortiori* in $\mathbf{I} + \Sigma^0_1\text{-DNS}_1$, which is consistent by Kleene's function-realizability, characteristic functions for all *negative* arithmetical relations (not involving $\vee$ or $\exists$) are unavoidable. In particular, $\mathbf{S} + \Sigma^0_1\text{-DNS}_1 \vdash \neg\neg\exists\beta\forall \mathrm{x}(\beta(\mathrm{x}) = 0 \leftrightarrow \forall \mathrm{y}\neg \mathrm{T}(\mathrm{x},\mathrm{x},\mathrm{y}))$, so $\mathbf{S} + \Sigma^0_1\text{-DNS}_1$ proves $\neg\neg\exists\alpha\neg\mathrm{GR}(\alpha)$ (equivalent by intuitionistic logic to $\neg\forall\alpha\neg\neg\mathrm{GR}(\alpha)$).

The fan theorem, which fails classically if only recursive sequences are allowed, holds for the arithmetical sequences. Every arithmetical relation has a classically equivalent negative translation. One might be tempted to propose $\Sigma^0_1\text{-DNS}_1$ as an "as yet undiscovered" axiom for intuitionistic analysis, guaranteeing the weak completeness of **HPC** for Beth semantics. Certainly $\mathbf{I} + \Sigma^0_1\text{-DNS}_1$ proves that nonrecursive sequences are unavoidable, and that the principle of testability for Σ^0_1 number-theoretic relations cannot be refuted.

7.3. **WMP, $\mathbf{MP}^\vee$, GDK and WKL with uniqueness.** WMP is provable in any subsystem of $\mathbf{I}$ extending $\mathbf{IA}_1$ and including $\forall\alpha\exists x$-continuous choice (cf. [25]), so MP_1 is equivalent to $\mathrm{MP}^\vee$ over $\mathbf{I}$.[23] It turns out that adding this bit of classical logic to $\mathbf{I}$ is equivalent to accepting a mathematical principle WKL!! which is stronger than the fan theorem but weaker constructively than WKL.

Berger and Ishihara [4] realized that adding the strong uniqueness hypothesis $\forall\alpha_{\mathrm{B}(\alpha)}\forall\beta_{\mathrm{B}(\beta)}[\exists \mathrm{x}\alpha(\mathrm{x}) \neq \beta(\mathrm{x}) \rightarrow \exists \mathrm{x}\rho(\overline{\alpha}(\mathrm{x})) \neq 0 \vee \exists \mathrm{x}\rho(\overline{\beta}(\mathrm{x})) \neq 0]$ to WKL results in a principle they called WKL!, which is interderivable with FT_1 over a weak constructive system; cf. Schwichtenberg [67]. Using instead the constructively weaker uniqueness hypothesis $\forall\alpha_{\mathrm{B}(\alpha)}\forall\beta_{\mathrm{B}(\beta)}[\forall \mathrm{x}\rho(\overline{\alpha}(\mathrm{x})) \neq 0 \ \&\ \forall \mathrm{x}\rho(\overline{\beta}(\mathrm{x})) \neq 0 \rightarrow \alpha = \beta]$, we obtain a stronger principle WKL!! which was introduced and analyzed in [59]. Part of the analysis involved arithmetical comprehension (arithmetical "unique choice" $\mathrm{AC}^{\mathrm{Ar}}_{00}!$), which holds in $\mathbf{S}$.

Theorem 4. WKL!! *has the following properties:*

(a) $\mathbf{IA_1}$ + WKL!! *proves* $\mathrm{MP}^\vee$, GDK, *and* $\neg\neg\mathrm{FT}_1$.
(b) WKL!! *is interderivable with* $\mathrm{MP}^\vee$ *over* $\mathbf{F}$ *augmented by arithmetical comprehension, and therefore also over* $\mathbf{S}$.
(c) WKL!! *is interderivable with* MP_1 *over* $\mathbf{I}$*, and over every subsystem of* $\mathbf{I}$ *which extends* $\mathbf{F}$ *and includes* $\forall\alpha\exists x$*-continuous choice.*

Proof. By the proofs of Theorems 3 and 4 in [59], $\mathbf{IA}_1$ + WKL!! entails $\mathrm{MP}^\vee$ and $\neg\neg$ WKL. The proof of Theorem 5 in [59] derived WKL!! from $\mathrm{MP}^\vee$ and $\neg\neg$ WKL in $\mathbf{IA}_1$ augmented by countable comprehension ("unique choice") for arithmetical relations. (a) and (b) follow by Theorem 1 in §4.2 above. Over $\mathbf{IA}_1$, continuous choice entails countable choice and countable comprehension, so (c) follows from (b) by Ishihara [24], [25]. □

From these results with the careful work in [14] it follows that the weak completeness of **HPC** for Beth semantics can be proved in $\mathbf{S}$ + WKL!! or $\mathbf{S}$ + GDK, or even in $\mathbf{F}$ + GDK augmented by arithmetical comprehension, using mathematical induction only over arithmetical relations. For strong completeness of **HPC** for Beth semantics, $\mathbf{I}$ + WKL!! would suffice.

[23] In [16] $\mathrm{MP}^\vee$ is identified as a restricted version of deMorgan's law and is renamed "Π^0_1-dML."

8. Epilogue

We began this essay with the conviction that Markov's Principle, Markov's Rule and the common notion of constructive proof are legitimate candidates for investigation in the spirit of Kreisel's ideal of informal rigour. Following Heyting's implicit advice, our preliminary investigation was limited to the context of formal systems for intuitionistic predicate logic, arithmetic and analysis.

However, informal rigour does not apply only to arguments formalizable in particular formal systems (cf. [54]). Brouwer stressed that to the extent formalization is justified at all, it must be preceded by the appropriate (informal intuitionistic) mathematics. We must remember that a literal translation of "metamathematics" is "after mathematics."

Intuitionistic (as opposed to classical) formal systems have nonderivable admissible rules, such as Church's Rule for **HA** and the Church-Kleene Rule for **I**. For corresponding closed instances of Markov's Rule and Markov's Principle, the first may hold for an intuitionistic system which does not prove the second. We could call a formal system with intuitionistic logic *Markovian* if it satisfies an appropriate version of Markov's Rule, and agree to use only arguments which can in principle be formalized in Markovian systems. Then we could confidently assert that every provably Δ^0_1 relation is recursive, without accepting MP_0 as an axiom (cf. [57]). In effect we would be substituting a broader, but equally precise, notion of constructive proof for the unimaginative "proof using intuitionistic logic."

Alternatively, $\mathrm{MP}_{\mathrm{QF}}$ could be added to intuitionistic arithmetic, guaranteeing that Δ^0_1 relations are recursive. MP_1 could be added to intuitionistic analysis or its neutral subsystem, without increasing the stock of provably recursive functions; then it would be possible to prove that the constructive arithmetical hierarchy does not collapse. GDK could be added to intuitionistic analysis as a way of asserting the adequacy of intuitionistic predicate logic for the intended interpretation, with possibly interesting mathematical consequences.

Intuitionistic mathematics is intended to evolve as new insights are attained. Kleene's decision to axiomatize **I** using only one classically false axiom which goes beyond Brouwer's published writings, Kripke's and Myhill's and Vesley's work inspired by Brouwer's creating subject arguments, Kreisel and Troelstra's work on lawless sequences and their projections, and the elaboration of Brouwer's ideas by Wim Veldman and others support this view. Surely Brouwer and Kreisel would have agreed that the notion of constructive proof is dynamic, evolving in time.

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