Models and Interpretations of (Intuitionistic) "Analysis"

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What is intuitionistic "analysis" and why should we care?

Historically, intuitionistic analysis (INT) was Brouwer's effort to understand constructively the structure of the continuum. He represented real numbers by Cauchy sequences of rationals, rejected arbitrary use of the law of excluded middle in logical reasoning, accepted full induction on the natural numbers ω and monotone bar induction on the "universal spread" ω^ω of all choice sequences, accepted countable and dependent choice . . . and then asserted that *every total function on the universe of choice sequences must be continuous.* This last step forced Brouwer to reject his own fixed point theorem and led to other bizarre conclusions, such as that the relation of inequality between real numbers does not satisfy the law of comparability.

On the other hand, Brouwer was able to give a constructive proof of e.g. the Jordan Curve Theorem, he had no problem using *reductio ad absurdum* to derive negative conclusions, and his proofs of existential assertions always provided (constructive approximations to) witnesses.

This last property was given primary importance by Errett Bishop, who developed a cautious constructivism (BISH) which neither condones nor violates Brouwer's continuity principle. Separately, the classical continuum, the intuitionistic continuum, and even the recursive continuum satisfy BISH, which has been described as "mathematics using intuitionistic logic." Brouwer and Bishop worked informally, leaving natural questions about consistency and relative independence for logicians to answer if they cared.

The point of this talk is to show how some of these consistency and independence questions have been answered for theories between two-sorted intuitionistic arithmetic $\mathbf{IA}_1$ and the intuitionistic formal system $\mathbf{I}$ of [Kleene and Vesley 1965], with some consequences for constructive analysis.

The tools which have been used for this purpose include

- classical models,
- Kripke models (S. Weinstein),
- realizability interpretations (S. Kleene, A. S. Troelstra),
- Gödel's negative interpretation (P. Krauss, R. Solovay),
- topological models (D. Scott, M. Krol, B. Scowcroft), and
- categorical models (see J. van Oosten's recent book).

This talk is problem-driven, emphasizing Kripke-Weinstein models and realizability, and omitting categories altogether.

Basic Constructive Function Existence Assumptions

Assumption (PR): *Primitive recursive functions* with arguments from $\mathbf{N} \cup \mathbf{N}^{\mathbf{N}}$ and values in $\mathbf{N}$ are constructive *in their parameters*.

Assumption (CF): Each *detachable subset* D of a set S has a characteristic function:

$$\forall s \in S(D(s) \vee \neg D(s)) \rightarrow \exists \chi \in \{0,1\}^S \forall s \in S(\chi(s) = 0 \leftrightarrow D(s)).$$

Assumption (AC!): If S and T are sets and $A \subseteq S \times T$ is *of functional character*, then A determines a function from S to T:

$$\forall s \in S\, \exists! t \in T\, A(s,t) \rightarrow \exists \varphi \in T^S\, \forall s \in S\, A(s, \varphi(s)),$$

where $\exists! t \in T\, A(s,t)$ expresses "there is exactly one $t \in T$ such that $A(s,t)$ holds."

For (CF) or (AC!), the χ or φ is unique, and is constructive in the parameters *relative to a justification of the hypothesis*.

How to formalize constructive and intuitionistic analysis?

Type-0 variables a, b, ..., m, n, ..., x, y, z, a_1, ... range over the natural numbers.

Prime formulas are of the form $\mathrm{s} = \mathrm{t}$ where s, t are *terms* of type 0.

The second sort of variables could range over arbitrary sets (intuitionistic "species") or over number-theoretic functions (intuitionistic "choice sequences"). But only *detachable* species $S \subseteq \omega$ satisfy $\forall n(n \in S \vee \neg n \in S)$, and most definable species are not detachable. Intuitionistic set theory is not really needed.

Kleene and Vesley used *type-1 variables*

$$\alpha, \beta, \gamma, \ldots, \alpha^1, \ldots$$

over sequences and let $\alpha = \beta$ abbreviate $\forall \mathrm{x}(\alpha(\mathrm{x}) = \beta(\mathrm{x}))$. Detachable species have characteristic functions, and properties of definable species $A(\alpha, x)$ are expressed by schemas.

$\mathbf{IA_1}$ is *two-sorted intuitionistic arithmetic*, formalized using finitely many primitive recursive function constants, parentheses denoting function application, and Church's λ binding a type-0 variable.

The mathematical axioms of $\mathbf{IA_1}$ are $(\mathrm{x} = \mathrm{y} \to \alpha(\mathrm{x}) = \alpha(\mathrm{y}))$, the defining equations for the function constants, the schema of mathematical induction for all formulas $\mathrm{A}(\mathrm{x})$, and the λ-reduction schema $(\lambda \mathrm{x}.\mathrm{u}(\mathrm{x}))\,(\mathrm{s}) = \mathrm{u}(\mathrm{s})$.

Easy Facts:

- $\mathbf{IA_1}$ proves that quantifier-free formulas (and formulas with only bounded number quantifiers) are decidable.
- $\mathbf{IA_1}$ proves the least number principle for decidable formulas.
- There is a classical model of $\mathbf{IA_1}$ in which the sequence variables range over the primitive recursive functions.

Notation: $\exists!$ denotes "there is exactly one," e.g. $\exists!\mathrm{y}\mathrm{B}(\mathrm{y})$ abbreviates $\exists\mathrm{y}\mathrm{B}(\mathrm{y})$ & $\forall\mathrm{x}\forall\mathrm{y}(\mathrm{B}(\mathrm{x})$ & $\mathrm{B}(\mathrm{y}) \to \mathrm{x} = \mathrm{y})$.

Minimal Analysis $\mathbf{M} = \mathbf{IA}_1 + \mathsf{AC}_{00}!$ where $\mathsf{AC}_{00}!$ is

$$\forall\mathrm{x}\exists!\mathrm{y}\mathrm{A}(\mathrm{x},\mathrm{y}) \to \exists\alpha\forall\mathrm{x}\mathrm{A}(\mathrm{x},\alpha(\mathrm{x})).$$

By intuitionistic logic with the decidability of number-theoretic equality, $\mathbf{M} \vdash \mathsf{AC}_{01}!$ where $\mathsf{AC}_{01}!$ is the comprehension schema

$$\forall\mathrm{x}\exists!\alpha\mathrm{A}(\mathrm{x},\alpha) \to \exists\beta\forall\mathrm{x}\mathrm{A}(\mathrm{x},(\beta)^{\mathrm{x}})$$

where $(\beta)^{\mathrm{x}}$ is $\lambda\mathrm{y}\beta(\langle\mathrm{x},\mathrm{y}\rangle)$ (the *x*th section of β).

M is strong enough to formalize the theory of recursive partial functions and functionals [Kleene 1969].

M also proves that *every detachable species of numbers has a characteristic function.* That is, **M** $\vdash$ CF_d where CF_d is the schema

$$\forall \mathrm{x}(\mathrm{A}(\mathrm{x}) \vee \neg \mathrm{A}(\mathrm{x})) \rightarrow \exists \rho \forall \mathrm{x}(\rho(\mathrm{x}) \leq 1 \ \& \ (\rho(\mathrm{x}) = 0 \leftrightarrow \mathrm{A}(\mathrm{x}))).$$

M $\vdash$ QF-AC$_{00}$ where QF-AC$_{00}$ is the restriction of the following schema to $\mathrm{A}(\mathrm{x}, \mathrm{y})$ quantifier-free or with bounded number quantifiers:

$$\forall \mathrm{x} \exists \mathrm{y} \mathrm{A}(\mathrm{x}, \mathrm{y}) \rightarrow \exists \alpha \forall \mathrm{x} \mathrm{A}(\mathrm{x}, \alpha(\mathrm{x})).$$

Theorem:

- (G. Vafeiadou) **M** $=$ **IA**$_1$ $+$ CF_d $+$ QF-AC$_{00}$.
- (GV) **IA**$_1$ $\nvdash$ QF-AC$_{00}$. (Classical model with $\alpha, \beta, \ldots$ ranging over all primitive recursive functions.)
- (GV) **IA**$_1$ $+$ QF-AC$_{00}$ $\nvdash$ CF_d. (Classical model of the general recursive functions.)
- **IA**$_1$ $+$ CF_d $\nvdash$ QF-AC$_{00}$. (Classical model of all functions with primitive recursive bounds.)

Countable choice, accepted by Brouwer and Bishop, can be expressed by adding the schema AC_{01}:

$$\forall \mathrm{x} \exists \alpha \mathrm{A}(\mathrm{x}, \alpha) \rightarrow \exists \beta \forall \mathrm{x} \mathrm{A}(\mathrm{x}, (\beta)^{\mathrm{x}}).$$

Because the classical least number principle fails intuitionistically, the general schema AC_{00}:

$$\forall \mathrm{x} \exists \mathrm{y} \mathrm{A}(\mathrm{x}, \mathrm{y}) \rightarrow \exists \alpha \forall \mathrm{x} \mathrm{A}(\mathrm{x}, \alpha(\mathrm{x}))$$

must also be considered as an intuitionistic choice principle.

Observations:

- Evidently $\mathbf{M} \subseteq \mathbf{IA}_1 + \mathsf{AC}_{00}$ since $\mathbf{IA}_1 + \mathsf{AC}_{00} \vdash \mathsf{AC}_{00}!$, and $\mathbf{M} + \mathsf{AC}_{00} \nvdash \mathsf{AC}_{01}$ even with classical logic.
- The question whether $\mathbf{M} \vdash \mathsf{AC}_{00}$ or $\mathbf{M} \nvdash \mathsf{AC}_{00}$ can't be settled by a classical model.

Theorem. (S. Weinstein) $\mathbf{M} \nvdash \mathrm{AC}_{00}$ (using a Kripke model).

Proof. A *canonical Kripke structure* for the language $\mathcal{L}$ of $\mathbf{M}$ is a structure $\mathcal{W} = \langle K, \leq, u_0, D_0, D_1, F_0, \dots, F_p, \Vdash \rangle$ where

- $u_0 \in K$, $\leq$ partially orders K, and $u_0 \leq v$ for all $v \in K$.
- $\omega \subseteq D_0(u) \subseteq D_0(v)$ for all $u \leq v \in K$.
- If $x \in D_0(u) \cap D_0(v)$ then $x \in D_0(w)$ for some $w \leq u, v$.
- $D_1 \subseteq \Delta_0^{\Delta_0}$ where $\Delta_0 = \cup_{u \in K} D_0(u)$.
- $\alpha(x) \in D_0(u)$ whenever $u \in K$, $x \in D_0(u)$ and $\alpha \in D_1$. Moreover, if $x \in \omega$ then $\alpha(x) \in \omega$ for all $\alpha \in D_1$.
- For $i \leq p$, F_i is a function extending the primitive recursive intended interpretation f_i of the function constant f_i of $\mathbf{HA}_1$ to $\Delta_0^{n_i} \times D_1^{m_i}$ in such a way that if $x_1, \dots, x_{n_i} \in D_0(u)$ and $\alpha_1, \dots, \alpha_{m_i} \in D_1$ then $F_i(x_1, \dots, x_{n_i}, \alpha_1, \dots, \alpha_{m_i})$ belongs to $D_0(u)$ and is independent of values $\alpha_j(y)$ for $y \notin D_0(u)$.

- $u \Vdash \mathrm{s} = \mathrm{t}$ iff $u \in K$ and s, t are terms of $\mathcal{L}[D_0(u), D_1]$ of type 0 such that $\mathrm{s} = \mathrm{t}$ is true when each $\mathrm{f_i}$ is interpreted by F_i. The forcing relation $u \Vdash \mathrm{A}$ extends to all sentences A of an infinitary language $\mathcal{L}[D_0(u), D_1]^+$, as follows.
- $u \Vdash (\mathrm{A} \rightarrow \mathrm{B})$ if for every $v \in K$ with $u \leq v$: if $v \Vdash \mathrm{A}$ then $v \Vdash \mathrm{B}$.
- $u \Vdash \forall \mathrm{x} \mathrm{A}(\mathrm{x})$ if for every $v \in K$ with $u \leq v$ and every $x \in D_0(v)$: $v \Vdash \mathrm{A}(\mathrm{x})$. Similarly for $\forall \alpha$.
- $u \Vdash \mathrm{A} \vee \mathrm{B}$ if $u \Vdash \mathrm{A}$ or $u \Vdash \mathrm{B}$, and similarly for &, $\exists$ and infinite conjunction and disjunction.

$\mathcal{W}$ is a *canonical model of* a theory $\mathbf{T}$ if $u_0 \Vdash \mathrm{E}$ for each closed theorem E of $\mathbf{T}$. A soundness and extended completeness theorem guarantees that for each closed formula E of $\mathcal{L}$:

$\vdash_{\mathbf{M}} \mathrm{E}$ if and only if $u_0 \Vdash \mathrm{E}$ for every canonical model $\mathcal{W}$ of $\mathbf{M}$.

Using the completeness theorem with a Smorynski-type collection operation on canonical Kripke models of **M**, Weinstein proved

Theorem. **M** has the *explicit definability property for sequences*: If $\mathrm{E}(\alpha)$ has only α free and $\vdash_{\mathbf{M}} \exists\alpha\mathrm{E}(\alpha)$, then for some $\mathrm{A}(\mathrm{x},\mathrm{y})$ with only x,y free:

$$\vdash_{\mathbf{M}} \forall\mathrm{x}\exists!\mathrm{y}\mathrm{A}(\mathrm{x},\mathrm{y}) \ \& \ \forall\alpha(\forall\mathrm{x}\mathrm{A}(\mathrm{x},\alpha(\mathrm{x})) \to \mathrm{E}(\alpha)).$$

Corollary. **M** $\not\vdash$ AC_{00}. There is a canonical model $\mathcal{W}_1$ of **M** such that

$u_0 \Vdash \forall\mathrm{x}\exists\mathrm{y}\neg\neg\mathrm{D}(\mathrm{x},\mathrm{y})$ but $u_0 \not\Vdash \exists\alpha\forall\mathrm{x}\neg\neg\mathrm{D}(\mathrm{x},\alpha(\mathrm{x}))$ where

$\mathrm{D}(\mathrm{x},\mathrm{y}) \equiv \mathrm{y} \leq 1 \ \& \ (\exists\mathrm{y}\mathrm{P}(\mathrm{x},\mathrm{y}) \to \mathrm{y} = 0) \ \& \ (\exists\mathrm{y}\mathrm{Q}(\mathrm{x},\mathrm{y}) \to \mathrm{y} = 1)$

where $\mathrm{P}(\mathrm{x},\mathrm{y})$ and $\mathrm{Q}(\mathrm{x},\mathrm{y})$ numeralwise express (in **M**) recursive properties $P(x,y), Q(x,y)$ such that $\{x : \exists y P(x,y)\}$ and $\{x : \exists y Q(x,y)\}$ are nonempty, disjoint and recursively inseparable.

Observation. Weinstein's proof also establishes that $\mathbf{M} \nvdash \mathrm{BC}_{00}$, where BC_{00} is *bounded countable choice*:

$$\forall x \exists y \leq \beta(x) A(x, y) \rightarrow \exists \alpha \forall x[\alpha(x) \leq \beta(x) \ \& \ A(x, \alpha(x))].$$

Apparently weaker than countable choice is the axiom schema of *boundedness* AB_{00}:

$$\forall x \exists y A(x, y) \rightarrow \exists \beta \forall x \exists y \leq \beta(x) A(x, y).$$

Observations:

- $\mathbf{IA}_1 + \mathrm{AB}_{00} + \mathrm{BC}_{00} = \mathbf{M} + \mathrm{AC}_{00}$.
- $\mathbf{IA}_1 + \mathrm{BC}_{00} \vdash \mathrm{CF}_{\mathrm{d}}$ but $\mathbf{IA}_1 + \mathrm{BC}_{00} \nvdash \mathrm{AB}_{00}$ (nor QF-AC_{00}) (classical model of primitive recursively bounded sequences).
- $\mathbf{IA}_1 + \mathrm{AB}_{00}$ proves that every Cauchy sequence of reals has a modulus of convergence (important for constructive analysis).

Three Formal Systems, from [Kleene-Vesley 1965]

Basic Analysis $\mathbf{B} = \mathbf{M} + \mathrm{AC}_{01} + \mathrm{BI}_1$ where BI_1 is the axiom schema of bar induction (w ranges over codes $\langle a_1, \ldots, a_k \rangle$ for finite sequences, $*$ denotes concatenation):

$$\forall\alpha\exists\mathrm{x}\rho(\overline{\alpha}(\mathrm{x})) = 0 \ \&\ \forall\mathrm{w}[\rho(\mathrm{w}) = 0 \rightarrow \mathrm{A}(\mathrm{w})]$$
$$\&\ \forall\mathrm{w}[\forall\mathrm{s}\mathrm{A}(\mathrm{w} * \langle\mathrm{s}\rangle) \rightarrow \mathrm{A}(\mathrm{w})] \rightarrow \mathrm{A}(\langle\,\rangle).$$

Intuitionistic Analysis $\mathbf{I} = \mathbf{B} + \mathrm{CC}_1$, where CC_1 is Kleene's algorithmic version of Brouwer's continuous choice principle:

$$\forall\alpha\exists\beta\mathrm{A}(\alpha,\beta) \rightarrow \exists\sigma\forall\alpha[\{\sigma\}[\alpha]\downarrow\ \&\ \forall\beta(\{\sigma\}[\alpha] = \beta \rightarrow \mathrm{A}(\alpha,\beta))].$$

Classical Analysis $\mathbf{C} = \mathbf{B} + (\mathrm{A} \vee \neg\mathrm{A})$.

Relative Consistency Theorem (Kleene): $\mathrm{Con}(\mathbf{B}) \Rightarrow \mathrm{Con}(\mathbf{I})$.

The proof used *recursive realizability* with sequences as realizing objects. Intuitively, ε *realizes-Ψ* E iff ε provides effective information validating E under the interpretation Ψ of its free variables Ψ. For example, ε *realizes-Ψ* $(\mathrm{A} \rightarrow \mathrm{B})$ iff for all α:
If α realizes-Ψ A then $\{\varepsilon\}[\alpha]$ is defined and realizes-Ψ B.
For prime formulas, realizability is equivalent to truth.

$\star$ *Main Theorem.* (Kleene)

(a) If $\Gamma \vdash_{\mathbf{I}} \mathrm{E}$ and all the formulas in Γ are realizable (by functions recursive in Δ), then E is realizable (by some ε recursive in Δ). Every extension of **I** by realizable axioms is simply consistent.

(b) Realizability can be formalized. To each formula E there is a formula $\varepsilon \mathbf{r} \mathrm{E}$ so that $\vdash_{\mathbf{M}} \neg\exists\varepsilon(\varepsilon\ \mathbf{r}\ (0 = 1))$ and for every formula E:
If $\vdash_{\mathbf{I}} \mathrm{E}$ then there is a p-functor u, expressing a recursive partial functional of only variables free in E, such that $\vdash_{\mathbf{B}} \mathrm{u}\downarrow\ \&\ (\mathrm{u}\,\mathbf{r}\,\mathrm{E})$ and so $\vdash_{\mathbf{B}} \exists\varepsilon\,(\varepsilon\,\mathbf{r}\,\mathrm{E})$. Therefore if **B** is simply consistent, so is **I**.

Definition. E is *self-realizing in* **T** if $\vdash_{\mathbf{T}} (\mathrm{E} \leftrightarrow \exists\varepsilon(\varepsilon \ \mathbf{r}\ \mathrm{E}))$.

Corollary. (Kleene) If E is self-realizing in **C** [in **B**] then

- If $\vdash_{\mathbf{I}} \mathrm{E}$ then E is classically [constructively] provable.
- If E is closed and $\vdash_{\mathbf{C}} \mathrm{E}$ then $\mathbf{I} + \mathrm{E}$ is consistent.

All arithmetical formulas are self-realizing even in **B**. So are

- The Gödel-Gentzen negative translations of all formulas.
- All *almost negative* formulas like $(\varepsilon \ \mathbf{r}\ \mathrm{E})$, which contain no $\vee$ and no $\exists$ except in subformulas $\exists\mathrm{x}(\mathrm{s} = \mathrm{t})$, $\exists\alpha(\mathrm{s} = \mathrm{t})$.
- All formulas in which the scope of each $\forall\alpha$ and each $\rightarrow$ is almost negative, such as Markov's Principle
  MP_1: $\forall\alpha(\neg\forall\mathrm{x}\neg\alpha(\mathrm{x}) = 0 \rightarrow \exists\mathrm{x}\alpha(\mathrm{x}) = 0)$.

All instances of double negation shift for numbers

$$\mathrm{DNS}_0:\quad \forall\mathrm{x}\neg\neg\mathrm{A}(\mathrm{x}) \rightarrow \neg\neg\forall\mathrm{x}\mathrm{A}(\mathrm{x})$$

are self-realizing in **C**, and so $\mathbf{I} + \mathrm{MP}_1 + \mathrm{DNS}_0$ is consistent.

Neither MP_1 nor DNS_0 is provable in **I**. (Use typed realizability.)

Gödel's negative translation $\mathrm{E} \mapsto \mathrm{E}^g$ is defined by

- $(\mathrm{s} = \mathrm{t})^g$ is $(\mathrm{s} = \mathrm{t})$.
- $(\mathrm{A} \vee \mathrm{B})^g$ is $\neg(\neg \mathrm{A}^g \& \neg \mathrm{B}^g)$.
- $\exists \mathrm{x} \mathrm{A}^g$ is $\neg \forall \mathrm{x} \neg \mathrm{A}^g$ and $\exists \alpha \mathrm{A}$ is $\neg \forall \alpha \neg \mathrm{A}^g$.
- The translation is transparent to $\rightarrow, \&, \neg$ and $\forall$.

Proposition: For every formula E:

- $\vdash_{\mathbf{C}} \mathrm{E} \leftrightarrow \mathrm{E}^g$ and $\vdash_{\mathbf{B}} \mathrm{E}^g \leftrightarrow \neg\neg \mathrm{E}^g$.
- If $\vdash_{\mathbf{C}} \mathrm{E}$ then E^g is a theorem of $\mathbf{B}$ + MP_1 + DNS_0, a proper subtheory of $\mathbf{C}$ consistent with $\mathbf{I}$.

In the second statement, DNS_0 can be slightly weakened to the schema AC_{01}°: $\forall \mathrm{x} \neg\neg \exists \alpha \mathrm{A}(\mathrm{x}, \alpha) \rightarrow \neg\neg \exists \beta \forall \mathrm{x} \neg\neg \mathrm{A}(\mathrm{x}, (\beta)^{\mathrm{x}})$ obtained from AC_{01} by replacing the constructive quantifiers $\forall$, $\exists$ by Krauss's classical quantifiers $\forall\neg\neg$ and $\neg\neg\exists$, respectively. So a natural equivalent of classical analysis is contained in $\mathbf{I}$.

A sharper result (unpublished) was obtained by R. Solovay. Call a formula *arithmetical* if it contains no type-1 quantifiers.

Theorem. (Solovay) Let **S** be the theory obtained from **M** + BI_1 by replacing the comprehension schema $\mathsf{AC}_{00}!$ by its restriction to arithmetical $\mathrm{A(x,y)}$. Let **BI** be the classical theory **S** + $(\mathrm{A} \vee \neg\mathrm{A})$. Then the negative interpretation of every theorem of **BI** is a theorem of **S** + MP_1.

Corollary: Let **A** = **M** + BI_1 + MP_1 (so **A** is a subtheory of **C** consistent with **I**). For every arithmetical formula $\mathrm{E(x,y,}\alpha)$:

$$\vdash_{\mathbf{A}} \neg\neg\exists\chi\forall\mathrm{x}[\chi(\mathrm{x}) = 0 \leftrightarrow \mathrm{E(x,y,}\alpha)].$$

In fact, **A** proves that every number-theoretic relation which is *classically* Δ^1_1 *definable* (perhaps with choice sequence parameters) cannot fail to have a characteristic function χ.

Kripke's Schema KS: $\exists\alpha[\mathrm{A} \leftrightarrow \exists \mathrm{x}\alpha(\mathrm{x}) = 0]$, inspired by Brouwer's theory of the creating subject, contradicts **I** (= **B** + CC_1).

An alternative to CC_1 is *continuous choice for numbers* CC_0:

$$\forall\alpha\exists \mathrm{x}\mathrm{A}(\alpha, \mathrm{x}) \rightarrow \exists\sigma\forall\alpha[\{\sigma\}(\alpha)\downarrow \ \& \ \forall \mathrm{x}(\{\sigma\}(\alpha) \simeq \mathrm{x} \rightarrow \mathrm{A}(\alpha, \mathrm{x}))]$$

or at least *continuous comprehension* $\mathsf{CC}_0!$:

$$\forall\alpha\exists! \mathrm{x}\mathrm{A}(\alpha, \mathrm{x}) \rightarrow \exists\sigma\forall\alpha[\{\sigma\}(\alpha)\downarrow \ \& \ \forall \mathrm{x}(\{\sigma\}(\alpha) \simeq \mathrm{x} \rightarrow \mathrm{A}(\alpha, \mathrm{x}))].$$

D. Scott's topological interpretation of the first-order intuitionistic theory of the real numbers with $<$ was easily transformed into a topological model of **B** + $\mathsf{CC}_0!$ + KS. M. Krol gave an example of CC_0 which failed in that model, and defined a more complex topological model of **B** + CC_0 + KS.

Corollary. $\mathbf{B} \subsetneq \mathbf{B} + \mathsf{CC}_0! = \mathbf{B} + \mathsf{CC}_1! \subsetneq \mathbf{B} + \mathsf{CC}_0 \subsetneq \mathbf{I}$.

Kleene's original realizability established

- If **T** is an extension of **B** by true, realizable axioms, then every closed prenex theorem of **T** is effectively (recursively) true.
- If E is closed and prenex, or the double negation of a closed prenex formula, then $\vdash_{\mathbf{I}} \mathrm{E} \Rightarrow \vdash_{\mathbf{B}} \mathrm{E}$.

Formalized q-realizability (realizability plus truth) established

Theorem. (Kleene)

- If $\vdash_{\mathbf{I}} \exists \mathrm{x} \mathrm{A}(\mathrm{x})$ where $\exists \mathrm{x} \mathrm{A}(\mathrm{x})$ is closed, then $\vdash_{\mathbf{I}} \mathrm{A}(\mathbf{n})$ for some numeral **n**.
- If $\vdash_{\mathbf{I}} \exists \alpha \mathrm{A}(\alpha)$ where $\exists \alpha \mathrm{A}(\alpha)$ is closed, then there is a gödel number *e* of a general recursive function φ such that $\vdash_{\mathbf{I}} \forall \mathrm{x}\{\mathbf{e}\} \downarrow \ \& \ \forall \alpha(\forall \mathrm{x}\{\mathbf{e}\}(\mathrm{x}) \simeq \alpha(\mathrm{x}) \rightarrow \mathrm{A}(\alpha))$.

The same proof works for **M**, **B**, **B** $+$ MP_1 $+$ DNS_0, and every extension **T** of **M** by recursively realizable axioms such as DC_1.

In “Classical desriptive set theory as a refinement of effective descriptive set theory,” APAL 2010, Yiannis Moschovakis adapted Kleene’s realizability to a language for descriptive set theory, explicitly coding Borel, analytic and co-analytic sets by sequences $\alpha \in \omega^\omega$; prime formulas are realizable if classically true. He used this realizability to analyze the (constructive) proof of Suslin’s Theorem and extract a proof of the Suslin-Kleene Theorem.

Main References:

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