Corrected Exercises in Reverse Constructive Analysis

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What is the point of reverse mathematics?

S. Simpson: The goal of **classical reverse mathematics** is to determine which set existence axioms are needed to prove a particular theorem of "ordinary" (classical) mathematics [**CLASS**].

D. Bridges: **Constructive reverse mathematics** asks

1. Which constructive principles are needed to prove particular theorems of Bishop's constructive mathematics [**BISH**]?
2. Which nonconstructive principles must be added to constructive mathematics in order to prove particular classical theorems?

W. Veldman: **Intuitionistic reverse mathematics** asks which intuitionistic axioms (countable choice AC, continuous choice CC, bar induction BI, Brouwer's fan theorem FT) are needed to prove a particular theorem of intuitionistic analysis [**INT**]. $>$

Russian reverse mathematics may ask which of the following axioms are needed to prove a particular theorem of [**RUSS**]:

- Church's Thesis CT_0: "Every constructive function is recursive"
- Markov's Principle MP_0: "If a recursive algorithm cannot fail to converge, then it converges"

For analysis, $\mathbf{BISH} \subseteq \mathbf{CLASS} \cap \mathbf{INT} \cap \mathbf{RUSS}$ but no two of CLASS, INT and RUSS are compatible.

Reverse mathematics must be formalizable. Reverse constructive analysis needs intuitionistic logic and precise mathematical axioms.

Two highly developed formal systems for intuitionistic analysis (Kleene and Vesley's **FIM**, Troelstra's **EL** + BI + CC) have been in use for decades, while BISH was developing informally. We seek a suitable formal framework for reverse constructive analysis.

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What are the objects of constructive mathematics?

E. Bishop: A **set** is defined by describing exactly what must be done in order to construct an element of the set and what must be done in order to show that two elements are equal.

It is *not* required that the equality relation be decidable, or that the set be enumerable.

Mathematical Folklore: A *function* φ is a rule or correspondence which assigns to each element x of a set S a unique element $\varphi(s)$ of a set T. (Notation: φ: $S \to T$.) The collection of all such φ is denoted by T^S, and considered to be a set.

Constructive Refinement: A **function** φ is a rule or correspondence which produces, for each element x (the *argument*) of a set S, a unique element $\varphi(x)$ (the *value*) in a set T.

An **extensional** function produces equal values when applied to equal arguments. T^S denotes the set of all extensional functions φ from S to T, with $\varphi = \psi \leftrightarrow \forall s \in S\,(\varphi(s) = \psi(s))$.

The natural numbers 0, 1, 2, ... constitute a set $\mathbf{N}$ with a **decidable** equality relation:

$$\forall m, n \in \mathbf{N}\, (m = n \vee \neg m = n).$$

A natural number n can represent the set of its predecessors, so $x \in n$ means $x < n$.

The Cartesian product $\mathbf{N} \times \mathbf{N}$ is the set of all pairs (m, n) of natural numbers, so if (m, n) is coded by a natural number such as $2^m \cdot 3^n$ then $\varphi : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ can be coded by an element of $\mathbf{N}^{\mathbf{N}}$.

Constructive analysis focuses on $\mathbf{N}^{\mathbf{N}}$ (Baire space), $\{0, 1\}^{\mathbf{N}}$ (Cantor space $2^{\mathbf{N}}$) and $\mathbf{R}$ (the constructive real numbers), each with a well-understood topology.

Working Hypothesis: The goal of reverse constructive analysis is to determine which *function* existence axioms are needed to prove a particular mathematical theorem about $\mathbf{N}$, $\mathbf{N}^{\mathbf{N}}$, $2^{\mathbf{N}}$, $\mathbf{R}$, $2^{\mathbf{R}}$, $\mathbf{N}^{\mathbf{R}}$, $\mathbf{R}^{\mathbf{R}}$, $\mathbf{R}^{\mathbf{N}}$, ... *using intuitionistic logic*.

Basic Constructive Function Existence Assumptions

Assumption (PR): *Primitive recursive functions* with arguments from $\mathbf{N} \cup \mathbf{N}^{\mathbf{N}}$ and values in $\mathbf{N}$ are constructive *in their parameters*.

Assumption (CF): Each *detachable subset* D of a set S has a characteristic function:

$\forall s \in S(D(s) \vee \neg D(s)) \rightarrow \exists \chi \in \{0,1\}^S \forall s \in S(\chi(s) = 0 \leftrightarrow D(s)).$

Assumption (AC!): If S and T are sets and $A \subseteq S \times T$ is *of functional character*, then A determines a function from S to T:

$$\forall s \in S \, \exists ! t \in T \, A(s,t) \rightarrow \exists \varphi \in T^S \, \forall s \in S \, A(s, \varphi(s)),$$

where $\exists ! t \in T \, A(s,t)$ abbreviates

$$\exists t \in T \, A(s,t) \; \& \; \forall x, t \in T \, (A(s,x) \; \& \; A(s,t) \rightarrow x = t).$$

For (CF) or (AC!), the χ or φ is unique, and is constructive in the parameters *relative to a justification of the hypothesis*.

Choice of Language, Logic and Minimal Axioms

Any well-founded reverse mathematics demands agreement on the language and logic to be used, and on a basic axiomatic theory. Two general principles expressible in the language are considered **equivalent** for this kind of mathematics, if each can be derived from (instances of) the other using the logic and basic axioms.

It is possible to formalize RUSS in the language of arithmetic, and BISH or INT in a two-sorted language – but only at the cost of arbitrary assumptions about the *representation* of functions with arguments in $\mathbf{N}^{\mathbf{N}}$. The natural statement of the Uniform Continuity Theorem needs a variable over functions from $\{0,1\}^{\mathbf{N}}$ to $\mathbf{N}$, but constructive analysis doesn't need all of $\mathbf{HA}^{\omega}$ or **CZF**.

A **three-sorted language** simplifies reverse constructive mathematics, relativization and comparison with CLASS.
The logic and basic axioms will prove the existence of continuous (in fact recursive) functions only.

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The **logic** is three-sorted intuitionistic logic with number-theoretic equality. Equality between functions is defined extensionally: $\alpha = \beta$ abbreviates $\forall x(\alpha(x) = \beta(x))$ and $F = G$ abbreviates $\forall\alpha(F(\alpha) = G(\alpha))$.

Terms s,t,... (of type 0), and *functors* u,v,... of type 1 and U,V,... of type 2 are defined from the variables and primitive recursive function constants using application and Church's λ. If U and v are functors and s is a term, then for example

- $\mathrm{U}[\mathrm{v}] + \mathrm{v}(\mathrm{s})$ is a term,
- $\lambda x.(\mathrm{U}[\mathrm{v}] + \mathrm{v}(x))$ is a functor of type 1, and
- $\lambda\alpha.(\mathrm{U}[\alpha] + \alpha(\mathrm{s}))$ is a functor of type 2.

If t is a term and x a number variable, we write $\mathrm{t}(x)$ for t, and t(s) for the result of substituting s for every free occurrence of x in t. There are two λ-**conversion axiom schemas**:

- $(\lambda x.\mathrm{t}(x))(\mathrm{s}) = \mathrm{t}(\mathrm{s})$, and
- $(\lambda\alpha.\mathrm{U}[\alpha])[\mathrm{v}] = \mathrm{U}[\mathrm{v}]$.

Restricted to the two-sorted language, the *minimal mathematical axioms* should be those of **EL** or the minimal theory $\mathbf{M}_1$ in which Kleene formalized recursion, both essentially based on two-sorted intuitionistic (Heyting) arithmetic $\mathbf{HA}_1$ with the mathematical induction schema

$$A(0) \wedge \forall x(A(x) \rightarrow A(x+1)) \rightarrow A(x)$$

for any formula $A(x)$. Both have axioms for primitive recursion and λ-reduction, but there is one important difference:

- **EL** assumes quantifier-free countable choice $\mathbf{qf\text{-}AC_{00}}$:

  $$\forall x \exists y A(x, y) \rightarrow \exists \alpha \forall x A(x, \alpha(x))$$

  where $A(x, y)$ is quantifier-free and has no free α.
- $\mathbf{M}_1$ assumes countable function comprehension $\mathbf{AC_{00}!}$:

  $$\forall x \exists ! y A(x, y) \rightarrow \exists \alpha \forall x A(x, \alpha(x))$$

  for every formula $A(x, y)$ in which α and x are free for y.

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Proposition 1. **qf-$\mathbf{AC_{00}}$!** can replace qf-AC_{00} in **EL**, since

(a) $\mathbf{HA_1}$ proves that quantifier-free formulas are decidable.

(b) $\mathbf{HA_1}$ proves, for every formula $B(y)$:

$$\forall y(B(y) \vee \neg B(y)) \mathbin{\&} \exists y B(y) \rightarrow \exists! y(B(y) \mathbin{\&} \forall z < y \neg B(z)).$$

Thus **EL** really doesn't assume any kind of countable choice.

Proposition 2.

(a) $\mathbf{HA_1}$ proves $\exists! y B(y) \rightarrow \forall y(B(y) \vee \neg B(y))$, but

(b) $\mathbf{HA_1}$ does *not* prove $\exists! \alpha B(\alpha) \rightarrow \forall \alpha(B(\alpha) \vee \neg B(\alpha))$ or even $\neg\neg(\exists! \alpha B(\alpha) \rightarrow \forall \alpha(B(\alpha) \vee \neg B(\alpha)))$.

Proposition 3. $\mathbf{M_1}$ proves

$\mathbf{AC_{01}!}$: $\quad \forall x \exists! \alpha A(x, \alpha) \rightarrow \exists \beta \forall x A(x, \lambda y.\beta(x, y))$,

where $A(x, \alpha)$ is any formula in which β and x are free for α, and $\beta(x, y)$ abbreviates $\beta(2^x \cdot 3^y)$.

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Proposition 4. $\mathbf{M}_1$ proves

$\mathbf{CF}_0$: $\quad \forall x(A(x) \vee \neg A(x)) \to \exists \chi \forall x(\chi(x) = 0 \leftrightarrow A(x))$,

where $A(x)$ is any formula.

Theorem 5. (gv)

(a) $\mathbf{EL}$ does *not* prove CF_0. That is, $\mathbf{EL}$ cannot prove that every detachable subset of $\mathbf{N}$ has a characteristic function.

(b) $\mathbf{EL} + \mathrm{CF}_0$ proves $\mathrm{AC}_{00}!$.

If $\mathbf{EL}^+$ is the definitional extension of $\mathbf{EL}$ obtained by adding the symbols and defining axioms for the finitely many constants of $\mathbf{M}_1$, then

(c) $\mathbf{EL}^+ + \mathrm{CF}_0$ is a conservative extension of $\mathbf{M}_1$.

(d) Every theorem of $\mathbf{EL}^+ + \mathrm{CF}_0$ is equivalent in $\mathbf{EL}^+ + \mathrm{CF}_0$ to a theorem of $\mathbf{M}_1$, by a uniform translation.

Observe that the theory $\mathbf{M}_1^-$ obtained from $\mathbf{M}_1$ by replacing $\mathrm{AC}_{00}!$ by qf-$\mathrm{AC}_{00}!$ (or qf-AC_{00}) has essentially the same strength as $\mathbf{EL}$.

Neither **EL** nor $\mathbf{M}_1$ proves the countable axiom of choice $\mathbf{AC}_{00}$ (like qf-AC_{00} but without the restriction to quantifier-free $A(x,y)$), as Scott Weinstein essentially showed in his PhD thesis.

The neutral basic theory **B** of Kleene and Vesley's *Foundations of Intuitionistic Mathematics* includes the countable choice schema

$\mathbf{AC}_{01}$: $\qquad \forall x \exists \alpha A(x,\alpha) \rightarrow \exists \beta \forall x A(x,(\lambda y.\beta(x,y))$.

instead of countable function comprehension $AC_{00}!$
B obviously proves AC_{00} (and $AC_{00}!$), so $\mathbf{M_1} \subsetneq \mathbf{B}$.

Although Brouwer and Bishop accepted countable choice, some constructivists (e.g. Fred Richman) doubt it. Reverse constructive analysis treats countable choice as an *optional* general function existence principle. Even if we believe countable choice, there is no harm in noting where it is needed and where it is not.

In addition to $\mathrm{AC}_{00}!$, from which $\mathrm{AC}_{01}!$ is derivable, our **three-sorted minimal theory $\mathbf{M}_2$** has a type-2 function comprehension axiom schema

$$\mathbf{AC}_{10}!: \qquad \forall\alpha\exists! m A(\alpha, m) \rightarrow \exists F \forall\alpha A(\alpha, F(\alpha)).$$

which guarantees the existence of all primitive recursive functions of type 2 and provides a characteristic function for each detachable subset of $\mathbf{N}^{\mathbf{N}}$. That is, $\mathbf{M}_2$ proves

$$\mathbf{CF}_1: \qquad \forall\alpha(A(\alpha) \vee \neg A(\alpha)) \rightarrow \exists H \forall\alpha(H(\alpha) = 0 \leftrightarrow A(\alpha)).$$

Proposition 6. (gv) Let $\mathbf{M}_2^-$ be the theory resulting from $\mathbf{M}_2$ by replacing $\mathrm{AC}_{10}!$ by qf-AC_{10} (or equivalently by qf-$\mathrm{AC}_{10}!$). Then

(a) $\mathbf{M}_2 = \mathbf{M}_2^- + \mathrm{CF}_1$.

(b) $\mathbf{HA}_2$ + qf-AC_{10} + CF_1 entails $\mathrm{AC}_{10}!$, where $\mathbf{HA}_2$ has symbols and axioms for all primitive recursive functions of type 2, with extensional equality.

Additional Constructive Axioms: The Fan Theorem

$\mathbf{N}^*$ is the tree of all finite sequences of natural numbers, with the empty sequence as root, and the predecessor relation determined by proper initial segments. A *spread* is a rooted subtree of $\mathbf{N}^*$ in which each node has at least one immediate successor; a *fan* is a spread in which only finite branching is allowed. Nodes are coded by natural numbers; a spread is coded by the characteristic function σ of the (detachable) set of its node codes.

Finite sequences of natural numbers are coded by *sequence numbers*, using primitive recursive functions lh, $\langle \ldots \rangle$, $\langle . \rangle_n$, $*$:

- $lh(\langle\,\rangle) = 0$ and $\langle\langle\,\rangle\rangle_0 = 0$.
- If $Seq(u)$ and $lh(u) > 0$ then $u = \langle\langle u\rangle_0, \ldots, \langle u\rangle_{lh(u)-1}\rangle$.
- If $Seq(u)$ and $Seq(v)$ then $u * v$ codes the concatenation of the sequences coded by u and v, in that order.
- If $\alpha \in \mathbf{N}^{\mathbf{N}}$ then $\overline{\alpha}(0) = \langle\,\rangle$ and $\overline{\alpha}(n+1) = \langle \alpha(0), \ldots, \alpha(n)\rangle$.

For ease of reading, we use u, v, w as metavariables ranging over the set **Seq** of codes for finite sequences. Let $u \in 2^*$ abbreviate $\forall n < lh(u) \, \langle u \rangle_n \leq 1$, and $\alpha \in 2^{\mathbf{N}}$ abbreviate $\forall x \alpha(x) \leq 1$.

INT accepts the Fan Theorem **(FT)** and the principle of monotone Bar Induction **(BI)**. Restricted versions of both are of interest for reverse constructive analysis.

The *full fan theorem* for the binary fan $\{0,1\}^{\mathbf{N}}$ is the schema

FT: $\quad \forall \alpha \in 2^{\mathbf{N}} \exists x A(\overline{\alpha}(x)) \rightarrow \exists y \forall \alpha \in 2^{\mathbf{N}} \exists x \leq y A(\overline{\alpha}(x))$.

Adding the hypothesis

$$\forall \mathrm{w}[\mathrm{w} \in 2^* \ \& \ \mathrm{A}(\mathrm{w}) \rightarrow \mathrm{A}(\mathrm{w} * \langle 0 \rangle) \ \& \ \mathrm{A}(\mathrm{w} * \langle 1 \rangle)]$$

to FT gives a schema $\mathbf{FT}_{\mathrm{mon}}$ which is constructively equivalent to FT, because the conclusion of the fan theorem is monotone and $\mathbf{HA}_1$ proves

$$\exists x A(\overline{\alpha}(x)) \leftrightarrow \exists y \exists u \exists v (\overline{\alpha}(y) = u * v \ \& \ A(u)).$$

The *decidable fan theorem* $\mathbf{FT}_d$ for the binary fan is the schema

$$\forall\alpha \in 2^{\mathbf{N}} \exists x\, A(\overline{\alpha}(x)) \;\&\; \forall w \in 2^{*}(A(w) \vee \neg A(w)) \\ \rightarrow \exists y \forall \alpha \in 2^{\mathbf{N}} \exists x \leq y\, A(\overline{\alpha}(x)).$$

Proposition 7. Over $\mathbf{HA}_1$, $\mathbf{EL}$ or $\mathbf{M}_1^-$, FT_d is interderivable with each of the schemas

(a) $\mathbf{FT}!$:

$$\forall\alpha \in 2^{\mathbf{N}} \exists! x\, A(\overline{\alpha}(x)) \rightarrow \exists y \forall \alpha \in 2^{\mathbf{N}} \exists x \leq y\, A(\overline{\alpha}(x))$$

(b) and $\mathbf{FT}_\mu$:

$$\forall\alpha \in 2^{\mathbf{N}} \exists^{\mu} x\, A(\overline{\alpha}(x)) \rightarrow \exists^{\mu} y \forall \alpha \in 2^{\mathbf{N}} \exists x \leq y\, A(\overline{\alpha}(x))$$

(where $\exists^{\mu} x B(x)$ abbreviates $\exists x(B(x) \;\&\; \forall y < x \neg B(y)))$

and entails the single axiom $\mathbf{FT}_1$:

$$\forall\alpha \in 2^{\mathbf{N}} \exists x\, \rho(\overline{\alpha}(x)) = 0 \rightarrow \exists y \forall \alpha \in 2^{\mathbf{N}} \exists x \leq y\, \rho(\overline{\alpha}(x)) = 0.$$

In $\mathbf{M}_1$ or $\mathbf{HA}_1 + \mathrm{CF}_0$, FT_1 is interderivable with FT_d.

Recently, Bishop constructivists have explored the reverse mathematics of two restricted versions, FT_c and $\mathrm{FT}_{\Pi^0_1}$, of the monotone fan theorem. They have derived FT_d from FT_c, and FT_c from $\mathrm{FT}_{\Pi^0_1}$, and have asked which of these consequence relations are strict.

Using CF_0 we can state each of their new versions efficiently as a single axiom with a free sequence variable. Thus $\mathbf{FT}_c$ becomes

$$\forall\alpha \in 2^{\mathbf{N}}\exists y\forall u \in 2^*\rho(\overline{\alpha}(y) * u)) = 0 \\ \rightarrow \exists y\forall\alpha \in 2^{\mathbf{N}}\forall z\rho(\overline{\alpha}(y + z)) = 0.$$

In the presence of CF_0, $\mathbf{FT}_{\Pi^0_1}$ can be stated

$$\forall w \in 2^*\forall m(\rho(w, m) = 0 \rightarrow \rho(w * \langle 0\rangle, m) = 0 \;\&\; \rho(w * \langle 1\rangle, m) = 0) \\ \&\; \forall\alpha \in 2^{\mathbf{N}}\exists y\forall n\,\rho(\overline{\alpha}(y), n) = 0 \\ \rightarrow \exists y\forall\alpha \in 2^{\mathbf{N}}\forall n\rho(\overline{\alpha}(y), n) = 0.$$

The Uniform Continuity Theorem for functions from $2^{\mathbf{N}}$ into $\mathbf{N}$ is naturally stated in the three-sorted language as **UCT(N)** :

$$\forall F \Big[\forall \alpha \in 2^{\mathbf{N}} \exists y \forall \beta \in 2^{\mathbf{N}} (\overline{\beta}(y) = \overline{\alpha}(y) \rightarrow F(\beta) = F(\alpha)) \\ \rightarrow \exists y (\forall \alpha \in 2^{\mathbf{N}} \forall \beta \in 2^{\mathbf{N}} (\overline{\beta}(y) = \overline{\alpha}(y) \rightarrow F(\beta) = F(\alpha)))\Big].$$

The definable version in the two-sorted language is $\mathbf{UCT}_{def}\mathbf{(N)}$:

$$\forall \alpha \in 2^{\mathbf{N}} (\exists ! x A(\alpha, x) \\ \& \ \exists y \forall \beta \in 2^{\mathbf{N}} (\overline{\beta}(y) = \overline{\alpha}(y) \rightarrow \forall x (A(\beta, x) \leftrightarrow A(\alpha, x)))) \\ \rightarrow \exists y \forall \alpha \in 2^{\mathbf{N}} \forall \beta \in 2^{\mathbf{N}} (\overline{\beta}(y) = \overline{\alpha}(y) \rightarrow \forall x (A(\beta, x) \leftrightarrow A(\alpha, x))).$$

Proposition 8. $\mathbf{M}_1 + \mathrm{UCT}_{def}(\mathbf{N})$ proves FT_d, and $\mathbf{M}_2 + \mathrm{UCT}(\mathbf{N})$ proves FT_d in the three-sorted language.

J. Berger has shown that FT_c and $\mathrm{UCT}(\mathbf{N})$ are interderivable, using only the resources of $\mathbf{M}_1$.

Additional Constructive Axioms: The Bar Theorem

$\mathbf{BI}_1$ is *"bar induction,"* a version of Brouwer's Bar Theorem, enabling backward induction on the universal spread:

$$\forall\alpha\exists x\rho(\overline{\alpha}(x)) = 0 \;\&\; \forall w[\rho(w) = 0 \rightarrow A(w)] \\ \;\&\; \forall w[\forall s A(w * \langle s\rangle) \rightarrow A(w)] \rightarrow A(\langle\,\rangle).$$

The *decidable bar induction* schema $\mathbf{BI}_d$ is

$$\forall\alpha\exists x R(\overline{\alpha}(x)) \;\&\; \forall w[R(w) \vee \neg R(w)] \;\&\; \forall w[R(w) \rightarrow A(w)] \\ \;\&\; \forall w[\forall s A(w * \langle s\rangle) \rightarrow A(w)] \rightarrow A(\langle\,\rangle).$$

Proposition 9.

(a) Over $\mathbf{HA}_1$, $\mathbf{EL}$ or $\mathbf{M}_1^-$, BI_d entails BI_1.

(b) In $\mathbf{M}_1$ or $\mathbf{HA}_1 + \mathrm{CF}_0$, BI_1 entails BI_d.

Monotone bar induction $\mathbf{BI}_{mon}$ is the schema

$$\forall\alpha\exists x R(\overline{\alpha}(x)) \;\&\; \forall w[R(w) \rightarrow A(w) \;\&\; \forall n R(w * \langle n\rangle)] \\ \;\&\; \forall w[\forall s A(w * \langle s\rangle) \rightarrow A(w)] \rightarrow A(\langle\,\rangle).$$

INT accepts BI_{mon}, which follows from BI_d or BI_1 using strong continuous choice CC_{10}. Full bar induction BI contradicts CC_{10}.

Some years ago Coquand showed that BI_d entails the principle of *Open Induction* OI_1 on Cantor space, and asked if the converse holds. Veldman recently explored the relationship between OI_1 and FT_d. Determining and separating the fundamental theories for reverse constructive mathematics is an adventure in progress.

This is a corrected version of the talk, with G. Vafeiadou's results unchanged.