INTUITIONISTIC MATHEMATICS AND LOGIC

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The first seeds of mathematical intuitionism germinated in Europe over a century ago in the constructive tendencies of Borel, Baire, Lebesgue, Poincaré, Kronecker and others. The flowering was the work of one man, Luitzen Egbertus Jan Brouwer, who taught mathematics at the University of Amsterdam from 1909 until 1951. By proving powerful theorems on topological invariants and fixed points of continuous mappings, Brouwer quickly built a mathematical reputation strong enough to support his revolutionary ideas about the nature of mathematical activity. These ideas influenced Hilbert and Gödel[1] and established intuitionistic logic and mathematics as subjects worthy of independent study.

Our aim is to describe the development of Brouwer's intuitionism, from his rejection of the classical law of excluded middle to his controversial theory of the continuum, with fundamental consequences for logic and mathematics. We borrow Kleene's formal axiomatic systems (incorporating earlier attempts by Kolmogorov, Glivenko, Heyting and Peano) for intuitionistic logic and arithmetic as subtheories of the corresponding classical theories, and sketch his use of gödel numbers of recursive functions to *realize* sentences of intuitionistic arithmetic including a form of Church's Thesis. Finally, we present Kleene and Vesley's axiomatic treatment of Brouwer's continuum, with the function-realizability interpretation which establishes its consistency.

1. Brouwer's Early Philosophy

In 1907 L. E. J. Brouwer published (in Dutch) his doctoral dissertation, whose title can be translated "On the Foundations of Mathematics."[2] This remarkable manifesto, with its heterodox views on mathematics, logic and language, critically examined and found fault with every major mathematical philosophy of the time. While Brouwer was aware of the work of the French intuitionist Poincaré and of Borel's constructive approach to the theory of sets, it was not in his nature to be a follower. In his 1912 inaugural address at the University of Amsterdam he referred to his own philosophy as *neo-intuitionism*,[3] but the vigor and creativity Brouwer brought to the subject over almost half a century of work have linked his name,

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[1]Brouwer's contribution to the dispute over the foundations of mathematics made a strong impression on all participants. In particular, his ideas strongly influenced those of his rival Hilbert, as well as the work of Gödel. For a full discussion of the history see [25].

[2]An English translation, from which the quotations in this section are taken, appears in [5].

[3]"Intuitionism and Formalism," an English translation of this address, appeared in the Bulletin of the American Mathematical Society in the same year, and is included in [1].

more than any other, with intuitionistic philosophy and mathematics. Many of the basic principles of his intuitionism were already clear in his dissertation.

In direct opposition to Russell and Whitehead's logicism, Brouwer asserted in 1907 that mathematics cannot be considered a part of logic. "Strictly speaking the construction of intuitive mathematics in itself is an *action* and not a *science*; it only becomes a science, i.e. a totality of causal sequences, repeatable in time, in a mathematics of the second order [metamathematics], which consists of the *mathematical consideration of mathematics* or *of the language of mathematics...* But there, as in the case of theoretical logic, we are concerned with an *application of mathematics*, that is, with an *experimental science*" ([5] p. 61).[4]

The discovery of non-Euclidean geometries showed, according to Brouwer, that Kant was only partly right in asserting that the intuitions of space and time are logically prior to (and independent of) experience. "...we can call a priori only that one thing which is common to all mathematics and is ... sufficient to build up all mathematics, namely the intuition of the many-oneness, the basic intuition of mathematics. And since in this intuition we become conscious of time as change per se, we can state: *the only a priori element in science is time*" ([5] p. 61).

Hilbert's formalist program was doomed to failure because "language ... is a means ... for the communication of mathematics but ... has nothing to do with mathematics" and is not essential for it. Moreover, the "... *existence* of a mathematical system satisfying a set of axioms can never be proved from the consistency of the logical system based on those axioms," but only by construction. "A fortiori it is not certain that any mathematical problem can either be solved or proved to be unsolvable" ([5] p. 79).

According to Brouwer, the paradoxes in set theory arise from the consideration of sets which are too large and abstract to be built up mathematically. Even Cantor's second number class (of the denumerably infinite ordinals) cannot exist, although the concept is consistent. Zermelo's proof of the wellordering principle from the axiom of choice is misguided. The continuum cannot be well-ordered, "firstly because the greater part of the elements of the continuum must be considered as unknown, and ... secondly because every well-ordered set is denumerable" ([5] pp. 84-85).

At the end of the dissertation, as the second of twenty-one "STATEMENTS (to be defended together with the thesis)," Brouwer asserts: "It is not only impossible to prove the admissibility of complete induction, but it ought neither to be considered as a special axiom nor as a special intuitive truth. Complete induction is an act of mathematical construction, justified simply by the basic intuition of mathematics" ([5] p. 98). This statement effectively dismisses the work of Peano, but admits a potential infinity of natural numbers with a method for showing that arbitrarily complicated properties (even those involving quantification over all natural numbers) hold for each. In this context Brouwer's earlier remark that "*all* or *every* ... tacitly involves the restriction: *insofar as belonging to a mathematical structure which is supposed to be constructed beforehand*" ([5] p. 76) suggests that the *structure* of the natural numbers can be understood as a completed construction even though the *collection* of all natural numbers cannot be surveyed at a glance.

[4]Within quotations all words in [ ] are ours, but the italics are Brouwer's own.

2. Intuitionistic Logic

One year after his dissertation, in "The unreliability of the logical principles," Brouwer argued against the use of classical logic in mathematics and science. He agreed with the principles of *syllogism* (if all A's are B, and all B's are C, then all A's are C) and *contradiction* (nothing is both A and not A), but not with the law of excluded middle (everything is A or not A) when it is applied to infinite systems.[5] In effect, Brouwer distinguished between the intuitionistically unacceptable $A \vee \neg A$ and the intuitionistically correct $\neg\neg(A \vee \neg A)$, making full use of the expressive power of the logical language to separate constructions which establish facts from constructions which establish consistency.[6]

Considering Brouwer's attitude toward formal logic, it is hardly surprising that he did not attempt to axiomatize intuitionistic reasoning. Nevertheless, he recognized the usefulness of formulating general principles which could be relied on for mathematical constructions. It is this which legitimizes formal systems for intuitionistic logic and mathematics, as long as each axiom and rule can be justified from an intuitionistic standpoint. As Kleene remarks on page 5 of [30], Brouwer's objection was only to formal reasoning without a corresponding (mathematical) meaning.

In 1925 Andrei Kolmogorov [32] proposed axioms for (minimal) intuitionistic reasoning with implication and negation only; his article, in Russian, attracted little or no attention in western Europe. Formal systems for intuitionistic propositional logic were published (in French) by Valerii Glivenko in 1928 ([15]) and 1929 ([16]). The first was incomplete. The second included two additional axioms suggested by Brouwer's student Arend Heyting, who presented his own detailed axiomatizations of intuitionistic propositional and predicate logic and a part of intuitionistic mathematics in three classic papers [19], [20], [21] the following year.[7] Heyting began with the caveat

> "Intuitionistic mathematics is a mental process, and every language, the formalistic one included, is an aid to communication only. It is in principle impossible to construct a system of formulas equivalent to intuitionistic mathematics, since the possibilities of thinking cannot be reduced to a finite number of rules constructed in advance."[8]

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[5][6] pp. 109-110. In fact, Brouwer asserted, "...the question of the validity of the principium tertii exclusi is equivalent to the question *whether unsolvable mathematical problems can exist.* There is not a shred of a proof for the conviction, which has sometimes been put forward, that there exist no unsolvable mathematical problems." We shall return to this question in Sections 3 and 4.

[6]D. Hesseling ([25] p.280) credits A. Kolmogorov [33] with the observation that for Brouwer a statement of the form $\neg B$ was *positive existential*: "there exists a chain of logical inferences, starting with the assumption that $[B]$ is correct and concluding with a contradiction" . By the same reasoning, $\neg\neg B$ asserts the consistency of B.

[7]Glivenko's 1929 note and the first part of Heyting's axiomatization were recently translated from the original French and German into English for the collection [36].

[8]An authoritative and readable summary of Heyting's contributions to the subject is A. S. Troelstra's [45], from which this smooth English translation of the introduction to Heyting [19] is borrowed.

Nevertheless, the effort to explain in metamathematical terms the difference between intuitionistic and classical logic made sense, because Brouwer's theory of the continuum contradicted classical logic.

In order to be acceptable intuitionistically, a general logical principle must be uniformly interpretable by means of constructions (proofs or computations). Kolmogorov's problem interpretation, and Heyting's association of assertions with their (constructively acceptable) proofs, are special cases of what has become known as the Brouwer-Heyting-Kolmogorov explication of the intuitionistic connectives and quantifiers.

Like Tarski's truth definition, the B-H-K interpretation is heuristic rather than mathematically precise, and is based on the assumption that true prime statements justify themselves. An implication can only be justified by a construction which uniformly converts any given justification of the hypothesis to a justification of the conclusion. A disjunction is justified by a construction which chooses a particular one of the disjuncts and provides its justification. The negation of a statement is justified by a construction which would convert any justification of the statement into a proof of a known contradiction. The intuitionistic objections to the classical laws of excluded middle and double negation can be explained on the basis of this interpretation.

Beginning around 1940 the American logician S. C. Kleene, who was sympathetic to intuitionistic ideas, devoted considerable effort to clarifying their precise relation to classical logic and mathematics.[9] The Hilbert-style formalisms for intuitionistic logic and arithmetic we present here are from Kleene [29] (p. 82), with minor changes in symbols and numbering. They represent an independent selection among many axioms proposed earlier by Kolmogorov, Glivenko, Heyting and Peano.

2.1. **The intuitionistic propositional calculus Pp.** The first step in the metamathematical study of any part of logic or mathematics is to specify a *formal language*. For propositional or sentential logic, the standard language has denumerably many distinct proposition letters $\mathrm{P}_0, \mathrm{P}_1, \mathrm{P}_2, \ldots$ and symbols $\&$, $\vee$, $\rightarrow$, $\neg$ for the propositional connectives "and," "or," "if ... then," and "not" respectively, with left and right parentheses (,) (sometimes written "[,]" for ease of reading). Classical logic actually needs only two connectives (since classical $\vee$ and $\rightarrow$ can be defined in terms of $\&$ and $\neg$), but the four intuitionistic connectives are independent. The classical language is thus properly contained in the intuitionistic, which is more expressive.

The most important tool of metamathematics is generalized induction, a method Brouwer endorsed. The class of (*well-formed*) *formulas* of the language of **Pp** is defined inductively by

(i) Each proposition letter is a (*prime*) *formula*.
(ii) If A, B are *formulas* so are $(\mathrm{A} \,\&\, \mathrm{B})$, $(\mathrm{A} \vee \mathrm{B})$, $(\mathrm{A} \rightarrow \mathrm{B})$ and $(\neg\mathrm{A})$.
(iii) Nothing is a *formula* except as required by (i) and (ii).

[9]Kleene was inspired by a remark of Hilbert and Bernays to consider "$\exists x A(x)$," for example, as "an incomplete communication, which is completed by giving an x such that $A(x)$ together with the further information required to complete the communication '$A(x)$' for that x." This interpretation led to recursive realizability; see §5.

As in classical logic, $(A \leftrightarrow B)$ abbreviates $((A \to B) \& (B \to A))$. Inessential parentheses are omitted on the convention that $\neg$ binds closer than $\&$, $\vee$ which bind closer than $\to$.

The building blocks for Kleene's version of intuitionistic propositional logic **Pp** are finitely many *axiom schemas*, each summarizing a potentially infinite collection of intuitionistically correct formulas, and one *rule of inference* expressing an intuitionistically acceptable principle of reasoning from hypotheses to a conclusion. The axioms are all formulas of the following forms:[10]

X1. $A \to (B \to A)$.
X2. $(A \to B) \to ((A \to (B \to C)) \to (A \to C))$.
X3. $A \to (B \to A \& B)$.
X4. $A \& B \to A$.
X5. $A \& B \to B$.
X6. $A \to A \vee B$.
X7. $B \to A \vee B$.
X8. $(A \to C) \to ((B \to C) \to (A \vee B \to C))$.
X9. $(A \to B) \to ((A \to \neg B) \to \neg A)$.
X10. $\neg A \to (A \to B)$.

The rule of inference of **Pp** is

R1 (*Modus Ponens*). From A and $A \to B$, conclude B.

A formal *proof* in **Pp** is any finite sequence $E_1, \ldots, E_k$ of formulas, each of which is an axiom or an immediate consequence, by the rule of inference, of two preceding formulas of the sequence.[11] Any proof is said to *prove* its last formula, which is therefore a *theorem* of **Pp**. We write $\vdash_{\mathbf{Pp}} E$ to denote that E is a theorem of **Pp**.

Example. Here is a complete formal proof (really a proof schema, as A may be any formula) in **Pp** of $\neg(A \& \neg A)$, indicating the reasons for each step.

(1) $(A \& \neg A) \to A$. [axiom by X4]
(2) $(A \& \neg A) \to \neg A$. [axiom by X5]
(3) $((A \& \neg A) \to A) \to (((A \& \neg A) \to \neg A) \to \neg(A \& \neg A))$. [axiom by X9]
(4) $((A \& \neg A) \to \neg A) \to \neg(A \& \neg A)$. [by R1 from (1), (3)]
(5) $\neg(A \& \neg A)$. [by R1 from (2), (4)]

If Γ is any collection of formulas and $E_1, \ldots, E_k$ any finite sequence of formulas each of which is a member of Γ, an axiom, or an immediate consequence by R1 of two preceding formulas, then $E_1, \ldots, E_k$ is a *derivation in* **Pp** *of* its last formula E_k *from* the assumptions Γ. We write $\Gamma \vdash_{\mathbf{Pp}} E$ to denote that such a derivation exists with $E_k = E$. The following theorem is proved by induction over the definition of a derivation; its converse follows from R1.

The Deduction Theorem. If Γ is any collection of formulas and A, B are any formulas such that $\Gamma \cup \{A\} \vdash_{\mathbf{Pp}} B$, then also $\Gamma \vdash_{\mathbf{Pp}} (A \to B)$.

[10]Glivenko's original axiom system consisted of X4 - X9 and variants of X2, X3. In [16] he added X1 and X10, which he attributed to Heyting. Heyting [19] had X1, X6, X9, X10 and variants of X2 and X8 as axioms but otherwise & and $\vee$ were treated differently. Heyting showed that his axioms are independent and do not prove $\neg\neg A \to A$. Kleene's version of X2 was designed to simplify the proof of the Deduction Theorem.

[11]This and similar descriptions abbreviate the obvious inductive definitions.

The axiomatization is designed so that classical propositional logic $\mathbf{Pp}^c$ results from $\mathbf{Pp}$ by strengthening the axiom schema X10 to

$\mathrm{X10}^c$. $\neg\neg \mathrm{A} \to \mathrm{A}$.

The definitions of *proof* and *derivation* for $\mathbf{Pp}^c$ are like those for $\mathbf{Pp}$ but with X1-$\mathrm{X10}^c$ instead of X1-X10. To show that $\mathbf{Pp}$ is a subtheory of $\mathbf{Pp}^c$ it suffices to prove $\vdash_{\mathbf{Pp}^c}$ X10, a relatively simple exercise.

In 1929 Glivenko [16] proved that if A is any formula such that $\vdash_{\mathbf{Pp}^c} \mathrm{A}$, then $\vdash_{\mathbf{Pp}} \neg\neg\mathrm{A}$. This simple form holds *only* for propositional logic, and is known as "Glivenko's Theorem."

Around 1933 Kurt Gödel [17] and Gerhard Gentzen (published posthumously in [14]) observed independently that $\mathbf{Pp}^c$ can be faithfully translated into $\mathbf{Pp}$.[12] Briefly, each proposition letter is replaced by its double negation, and then $\vee$ is replaced inductively by its classical definition in terms of $\neg$ and &. If Γ^g, A^g are the translations of Γ, A respectively then

(i) $\vdash_{\mathbf{Pp}^c} (\mathrm{A}^g \leftrightarrow \mathrm{A})$, and
(ii) $\Gamma^g \vdash_{\mathbf{Pp}} \mathrm{A}^g$ if and only if $\Gamma \vdash_{\mathbf{Pp}^c} \mathrm{A}$.

In 1934-35 Gentzen [13] proved a normal form theorem for an intuitionistic sequent calculus, giving an effective algorithm for deciding whether an arbitrary formula A is or is not provable in $\mathbf{Pp}$. Since intuitionistic propositional logic has no finite truth-table interpretation, the decision algorithm for $\mathbf{Pp}$ is more complicated than for $\mathbf{Pp}^c$.

2.2. **The intuitionistic first-order predicate calculus Pd.** The pure first-order language has individual variables $\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3, \ldots$, and countably infinitely many distinct predicate letters $\mathrm{P}_1(\ldots), \mathrm{P}_2(\ldots), \mathrm{P}_3(\ldots), \ldots$ of arity n for each $n = 0, 1, 2, \ldots$, including the 0-ary proposition letters. There are two new logical symbols $\forall$ ("for all") and $\exists$ ("there exists").

The *terms* of the language of $\mathbf{Pd}$ are the individual variables. The *formulas* are defined by

(i) If $\mathrm{P}(\ldots)$ is an n-ary predicate letter and $\mathrm{t}_1, \ldots, \mathrm{t}_n$ are terms then $\mathrm{P}(\mathrm{t}_1, \ldots, \mathrm{t}_n)$ is a (*prime*) *formula*.
(ii) If A, B are *formulas* so are (A & B), $(\mathrm{A} \vee \mathrm{B})$, $(\mathrm{A} \to \mathrm{B})$ and $(\neg\mathrm{A})$.
(iii) If A is a *formula* and x an individual variable, then $(\forall \mathrm{x}\mathrm{A})$ and $(\exists \mathrm{x}\mathrm{A})$ are *formulas*.
(iv) Nothing else is a *formula*.

We use $\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{w}, \mathrm{x}_1, \mathrm{y}_1, \ldots$ and $\mathrm{A}, \mathrm{B}, \mathrm{C}, \ldots, \mathrm{A}(\mathrm{x}), \mathrm{A}(\mathrm{x}, \mathrm{y}), \ldots$ as metavariables for variables and formulas, respectively. Anticipating applications (e.g. to arithmetic), $\mathrm{s}, \mathrm{t}, \mathrm{s}_1, \mathrm{t}_1, \ldots$ vary over terms. In omitting parentheses, $\forall\mathrm{x}$ and $\exists\mathrm{x}$ are treated like $\neg$. The *scope* of a quantifier, and *free* and *bound* occurrences of a variable in a formula, are defined as usual. A formula in which every variable is bound is a *sentence* or *closed formula*.

If x is a variable, t a term, and A(x) a formula which may or may not contain x free, then A(t) denotes the result of substituting an occurrence of t for each free

[12]In 1925 Kolmogorov [32] published a different negative translation for a fragment with $\to$ and $\neg$ only.

occurrence of x in A(x). The substitution is *free* if no free occurrence in t of any variable becomes bound in A(t); in this case we say t *is free for* x *in* A(x).[13]

In addition to X1 - X10, **Pd** has two new *axiom schemas*, where A(x) may be any formula and t any term free for x in A(x):

X11. $\forall$xA(x) $\rightarrow$ A(t).
X12. A(t) $\rightarrow$ $\exists$xA(x).

In addition to R1, **Pd** has two new *rules of inference*:

R2. From C $\rightarrow$ A(x) where x does not occur free in C, conclude C $\rightarrow$ $\forall$xA(x).
R3. From A(x) $\rightarrow$ C where x does not occur free in C, conclude $\exists$xA(x) $\rightarrow$ C.

A *deduction* (or *derivation*) in **Pd** *of* a formula E *from* a collection Γ of assumption formulas is a finite sequence of formulas, each of which is an axiom by X1 - X12, or a member of Γ, or follows immediately by R1, R2 or R3 from one or two formulas occurring earlier in the sequence. A *proof* is a deduction from no assumptions.

If Γ is a collection of sentences and E a formula, the notation $\Gamma \vdash_{\mathbf{Pd}}$ E means that a deduction of E from Γ exists. If Γ is a collection of formulas, we write $\Gamma \vdash_{\mathbf{Pd}}$ E only if there is a deduction of E from Γ in which neither R2 nor R3 is used with respect to any variable free in Γ. With this restriction, the deduction theorem extends to **Pd**: If $\Gamma \cup \{A\} \vdash_{\mathbf{Pd}}$ B then $\Gamma \vdash_{\mathbf{Pd}}$ (A $\rightarrow$ B).

Example. Here is a deduction in **Pd** of $\exists$xA(x) from $\forall$xA(x) without using R2 or R3:

(1) $\forall$xA(x) $\rightarrow$ A(x). [axiom by X11, with x free for x in A(x)]
(2) $\forall$xA(x). [hypothesis]
(3) A(x). [by R1 from (1) and (2)]
(4) A(x) $\rightarrow$ $\exists$xA(x). [axiom by X12, with x free for x in A(x)]
(5) $\exists$xA(x). [by R1 from (3) and (4)]

Then $\vdash_{\mathbf{Pd}}$ $\forall$xA(x) $\rightarrow$ $\exists$xA(x) follows by the deduction theorem.

Classical predicate logic $\mathbf{Pd}^c$ comes from **Pd** by strengthening X10 to X10^c. The negative interpretation extends to predicate logic using the classical definition of $\exists$ in terms of $\forall$ and $\neg$. The difference between constructive and classical proofs of existence is starkly illustrated by the strong *existence* and *disjunction* properties of **Pd**:

- If $\vdash_{\mathbf{Pd}}$ $\exists$xA(x) where no variable other than x is free in A(x), then $\vdash_{\mathbf{Pd}}$ A(x) and hence $\vdash_{\mathbf{Pd}}$ $\forall$xA(x).
- If $\vdash_{\mathbf{Pd}}$ $\forall$x[A(x) $\vee$ B(x)] where no variable other than x is free in A(x) or B(x), then $\vdash_{\mathbf{Pd}}$ $\forall$xA(x) or $\vdash_{\mathbf{Pd}}$ $\forall$xB(x).

2.3. **Intuitionistic Predicate Logic with Equality Pd[=].** To be useful for mathematics, the formal language must contain a binary predicate constant $\cdot = \cdot$ denoting equality. If s, t are any terms then s = t is a *prime formula* in which all the variables free in s or t are free. Every prime formula of the language of **Pd** is also prime in **Pd**[=], and the *formulas* are built up from the prime formulas using $\&, \vee, \rightarrow, \neg, \forall$ and $\exists$ as before.

[13]It matters which formula was originally designated by the notation A(x), since if x, y are distinct and both occur free in A(x) then the sequence of free substitutions x $\mapsto$ y $\mapsto$ x results in a formula different from A(x).

The axioms of **Pd**[=] are all formulas of the extended language of the forms X1 - X12, together with the following equality axioms, where x, y and z are distinct variables and A(x) may be any prime formula of the language of **Pd** in which y is free for x.

XE1. $x = y \rightarrow (x = z \rightarrow y = z)$.
XE2. $x = x$.
XE3. $x = y \rightarrow (A(x) \rightarrow A(y))$.

The rules of inference are R1 - R3 extended to the new language. The *replacement property of equality* holds: If A(x) is any formula, and x and y are variables such that y is free for x in A(x) and does not occur free in A(x) (unless y is x), then $\vdash_{\mathbf{Pd}[=]} \; x = y \rightarrow (A(x) \leftrightarrow A(y))$.

The axioms guarantee that = is an equivalence relation but do not prove that = is decidable or even *stable* under double negation, since $\nvdash_{\mathbf{Pd}[=]} \; \neg\neg(x = y) \rightarrow (x = y)$.[14]

A typical mathematical application will have individual constants and function symbols, with an appropriate definition of *term*, and all prime formulas will be equations. Then the equality axioms for the function constants will take the place of XE3, and XE2 may follow from the mathematical axioms.

3. Intuitionistic Arithmetic HA

Heyting [20] first axiomatized intuitionistic arithmetic, which is called "Heyting arithmetic" in his honor.[15] Kleene's version [29] (p. 82) of **HA** has constants and axioms for zero, successor, addition and multiplication, and the unrestricted axiom schema of mathematical induction.

3.1. Heyting arithmetic and Peano arithmetic.

The language of **HA** is an applied version of the language of **Pd**[=], with no predicate letters, but with an individual constant **0** and function constants $'$, $+$ and $\cdot$. *Terms* are defined inductively:

(i) **0** is a *term*.
(ii) Each individual variable is a *term*.
(iii) If s is a *term*, so is (s').
(iv) If s and t are *terms*, so are $(s + t)$ and $(s \cdot t)$.
(v) Nothing is a *term* except as required by (i) - (iv).

The *prime formulas* are the expressions of the form $(s = t)$ where s and t are terms. *Formulas* are built up from these as usual, omitting parentheses on the convention that $'$ binds closer than $+, \cdot$. Every occurrence of an individual variable in a term t is free in t (and in every prime formula containing t). A term or formula without free variables is *closed*.

[14] Heyting [20] introduced three distinct symbols to express three kinds of equality relations: [intensional] identity, mathematical identity, and defined equality. For arithmetic all three notions coincide, and number-theoretic equality is decidable. Equality of choice sequences, to be discussed in the section on Brouwer's continuum, is defined extensionally and is stable under double negation but undecidable.

[15] J. van Oosten [41] points out that the formalizations of **HA** as a subtheory of Peano arithmetic owe as much to Gödel [17] and to Kleene [29] as to Heyting. Hesseling [25] observes that Gödel used Herbrand's axioms.

The *axioms* of **HA** are the schemas X1 - X12 for the language of arithmetic, the equality axiom XE1, the axiom schema of *mathematical induction* for arbitrary formulas A(x):

XInd. $\mathrm{A}(\mathbf{0})\ \&\ \forall \mathrm{x}(\mathrm{A}(\mathrm{x}) \rightarrow \mathrm{A}(\mathrm{x}')) \rightarrow \forall \mathrm{x}\mathrm{A}(\mathrm{x})$,

and the additional axioms for the primitive recursive function constants:

XN1. $\mathrm{x} = \mathrm{y} \rightarrow \mathrm{x}' = \mathrm{y}'$.
XN2. $\mathrm{x}' = \mathrm{y}' \rightarrow \mathrm{x} = \mathrm{y}$.
XN3. $\neg(\mathrm{x}' = \mathbf{0})$.
XN4. $\mathrm{x} + \mathbf{0} = \mathrm{x}$.
XN5. $\mathrm{x} + (\mathrm{y}') = (\mathrm{x} + \mathrm{y})'$.
XN6. $\mathrm{x} \cdot \mathbf{0} = \mathbf{0}$.
XN7. $\mathrm{x} \cdot (\mathrm{y}') = (\mathrm{x} \cdot \mathrm{y}) + \mathrm{x}$.

The *rules of inference* of **HA** are R1 - R3 for the language of arithmetic. *Derivations* and *proofs* are defined inductively as usual, and $\Gamma \vdash_{\mathbf{HA}} \mathrm{E}$ means that a derivation in **HA** of E from Γ exists in which neither R2 nor R3 is used with respect to a variable free in Γ.

The construction of the (standard) natural numbers generates a name or *numeral* for each, thus $\mathbf{0}''$ is the numeral for 2. Each closed term t of the language expresses (under the intended interpretation) a particular natural number; if $\mathbf{t}$ is the corresponding numeral then $\vdash_{\mathbf{HA}} \mathrm{t} = \mathbf{t}$. The natural numbers are discrete:

- $\vdash_{\mathbf{HA}}\ (\mathrm{x} = \mathrm{y}) \vee \neg(\mathrm{x} = \mathrm{y})$.
- $\vdash_{\mathbf{HA}}\ \neg\neg(\mathrm{x} = \mathrm{y}) \rightarrow (\mathrm{x} = \mathrm{y})$.
- $\vdash_{\mathbf{HA}}\ (\mathrm{x} = 0) \vee \exists \mathrm{y}(\mathrm{x} = \mathrm{y}')$.

Classical Peano arithmetic **PA** comes from **HA** by strengthening X10 to $\mathrm{X10}^c$. Gödel [17] extended the negative translation to **PA**, reducing the consistency of **PA** to that of **HA** and showing that **HA** cannot prove its own consistency.

Kleene's proofs in [29] of Gödel's first and second incompleteness theorems apply to **HA** as well as to **PA**. The arithmetization of metamathematics was carried out in the intuitionistic subsystem, so every primitive recursive predicate can be numeralwise expressed in **HA** by a *decidable* formula. If $\mathrm{T}(\mathrm{e}, \mathrm{x}, \mathrm{w})$ is such a formula numeralwise expressing "w is the gödel number of a computation of $\{e\}(x)$," and A(x) is $\exists \mathrm{z}\mathrm{T}(\mathrm{x}, \mathrm{x}, \mathrm{z})$, then $\not\vdash_{\mathbf{HA}} \neg\neg\forall \mathrm{x}(\mathrm{A}(\mathrm{x}) \vee \neg\mathrm{A}(\mathrm{x}))$ and so $\mathbf{HA} + \neg\forall \mathrm{x}(\mathrm{A}(\mathrm{x}) \vee \neg\mathrm{A}(\mathrm{x}))$ is consistent.[16] This is a special case of the remarkable fact that intuitionistic arithmetic is consistent with a classically false form of Church's Thesis, as we are about to see.

3.2. Kleene's recursive realizability for intuitionistic arithmetic.

The origin and development of recursive realizability are delightfully recounted by Jaap van Oosten in [41]. Stephen Kleene, a student of Alonzo Church, took seriously Hilbert and Bernays' assertion in [24] that "a statement of the form 'there exists a number n with property $A(n)$' is ... an incomplete rendering of a more precisely determined proposition, which consists either in directly giving a number n with the property $A(n)$, or providing a procedure by which such a number can be found ...." Kleene generalized this idea to interpret every compound sentence of intuitionistic arithmetic as an incomplete communication of an effective procedure by

[16]To show $\neg\mathrm{E}$ is consistent with an intuitionistic system one must show that $\neg\neg\mathrm{E}$ (not just E) is unprovable.

which its correctness might be established, and then applied Church's Thesis to identify "effective" with "recursive." The result was 1945-*realizability* or number-realizability.[17]

For the inductive definition, n and m range over natural numbers, and $(n)_i$ is the exponent of the i^{th} prime in the complete prime factorization of n (counting 2 as the 0^{th} prime, and setting $(0)_i = 0$ by convention. Ordered pairs are coded by $(n, m) = 2^n \cdot 3^m$, and $\{n\}(m)$ denotes the result of applying the n^{th} recursive partial function to the argument m.

Definition. (Kleene 1945) A number n *realizes* a sentence E only as follows:

1. n *realizes* a closed prime formula $\mathrm{r} = \mathrm{t}$, if $\mathrm{r} = \mathrm{t}$ is true under the intended interpretation.
2. n *realizes* A & B, if $(n)_0$ *realizes* A and $(n)_1$ realizes B.
3. n *realizes* $\mathrm{A} \vee \mathrm{B}$, if either $(n)_0 = 0$ and $(n)_1$ *realizes* A, or $(n)_0 \neq 0$ and $(n)_1$ *realizes* B.
4. n *realizes* $\mathrm{A} \to \mathrm{B}$, if, for every m: if m *realizes* A then $\{n\}(m)$ is defined and *realizes* B.
5. n *realizes* $\neg\mathrm{A}$, if no m *realizes* A.
6. n *realizes* $\forall \mathrm{x} \mathrm{A}(\mathrm{x})$, if, for every m: $\{n\}(m)$ is defined and *realizes* $\mathrm{A}(\mathbf{m})$.
7. n *realizes* $\exists \mathrm{x} \mathrm{A}(\mathrm{x})$, if $(n)_1$ *realizes* $\mathrm{A}(\mathbf{m})$ where $m = (n)_0$.

A formula E is *realizable* if some n realizes the universal closure $\forall$E of E.

Kleene conjectured that every closed theorem of **HA** was realizable. His student David Nelson verified the conjecture, then formalized the proof in an extension of **HA**.[18] The key lemma states that for each closed term t expressing the number corresponding to the numeral **t**:

$$n \text{ realizes } \mathrm{A}(\mathrm{t}) \;\Leftrightarrow\; n \text{ realizes } \mathrm{A}(\mathbf{t}).$$

Nelson's Theorem. If $\mathrm{C}_1, \ldots, \mathrm{C}_k \vdash_{\mathbf{HA}} \mathrm{A}$ and $\mathrm{C}_1, \ldots, \mathrm{C}_k$ are realizable, so is A.

Corollary 1. If $\vdash_{\mathbf{HA}} \forall \mathrm{x}_1 \ldots \forall \mathrm{x}_n \exists \mathrm{y} \mathrm{A}(\mathrm{x}_1, \ldots, \mathrm{x}_n, \mathrm{y})$ where $\mathrm{A}(\mathrm{x}_1, \ldots, \mathrm{x}_n, \mathrm{y})$ contains free only $\mathrm{x}_1, \ldots, \mathrm{x}_n, \mathrm{y}$ then there is a general recursive function ψ of n variables such that for all values of $\mathrm{x}_1, \ldots, \mathrm{x}_n$: If $\psi(x_1, \ldots, x_n) = y$, then $\mathrm{A}(\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{y})$ is realizable.

In [28] Kleene observed that the cases of the definition for $\vee$, $\to$ and $\exists$ could be modified to give another notion (later called *realizable*-$\vdash$) for which the analogue of Nelson's Theorem held, with the following result.[19]

Corollary 2 (to the version of Nelson's Theorem for "realizable-$\vdash$"). Assuming $\vdash_{\mathbf{HA}} \forall \mathrm{x}_1 \ldots \forall \mathrm{x}_n \exists \mathrm{y} \mathrm{A}(\mathrm{x}_1, \ldots, \mathrm{x}_n, \mathrm{y})$ where $\mathrm{A}(\mathrm{x}_1, \ldots, \mathrm{x}_n, \mathrm{y})$ contains free only $\mathrm{x}_1, \ldots, \mathrm{x}_n, \mathrm{y}$, there is a general recursive function ψ such that for all values of $x_1, \ldots, x_n$:

$$\vdash_{\mathbf{HA}} \mathrm{A}(\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{y}) \text{ where } \psi(x_1, \ldots, x_n) = y.$$

[17] Kolmogorov [33] had earlier proposed a "problem interpretation" of the intuitionistic connectives, but had not connected it with recursive functions.

[18] The technical work of formalization, in [40], was nontrivial. Kleene [28] announced and interpreted Nelson's results. For a comprehensive and comprehensible modern treatment, see Troelstra [48].

[19] The existence and disjunction properties for **HA** were implicit special cases, as Kleene later noted.

3.3. **Church's Thesis.** It is possible to express Church's Thesis in the language of arithmetic. One version (which includes countable choice) is the schema CT_0:

$$\forall x \exists y A(x,y) \rightarrow \exists e \forall x \exists w[T(e,x,w) \ \& \ A(x,U(w))]$$

where $T(e,x,w)$ (numeralwise) expresses "w is the gödel number of a computation of $\{e\}(x)$," and $A(x,U(w))$ abbreviates $\forall z(U(w,z) \rightarrow A(x,z))$ where $U(w,z)$ (numeralwise) expresses "z is the value, if any, computed by the computation with gödel number w; otherwise $z = 0$". The gödel numbering is primitive recursive, and $T(e,x,w)$ and $U(w,z)$ are quantifier-free.[20] Nelson's Theorem entails the following consistency and independence results.

Corollary 3. **HA** + CT_0 is consistent.

Corollary 4. If $A(x)$ is $\exists y T(x,x,y)$ and $B(x)$ is $A(x) \vee \neg A(x)$, then

(i) $\nvdash_{\mathbf{HA}} \forall x(A(x) \ \vee \ \neg A(x))$,
(ii) $\nvdash_{\mathbf{HA}} \neg\neg\forall x(A(x) \ \vee \ \neg A(x))$, and
(iii $\nvdash_{\mathbf{HA}} \forall x \neg\neg B(x) \rightarrow \neg\neg\forall x B(x)$.

While **HA** is a subsystem of **PA**, **HA** + CT_0 is a nonclassical arithmetic. To see why, let $A(x,y)$ be $(y = 0 \rightarrow \forall z \neg T(x,x,z))$ & $(y \neq 0 \rightarrow T(x,x,y \dot{-} 1))$. Evidently $\vdash_{\mathbf{PA}} \forall x \exists y A(x,y)$. In **HA** the hypothesis $\forall x \exists w[T(e,x,w) \ \& \ A(x,U(w))]$ implies $\exists w[T(e,e,w) \ \& \ A(e,U(w))]$, but $\vdash_{\mathbf{HA}} \forall w[T(e,e,w) \rightarrow A(e,w+1)]$ and the gödel numbering satisfies $\vdash_{\mathbf{HA}} \forall w(U(w) \leq w)$ and $\vdash_{\mathbf{HA}} \forall u \forall v[A(e,u) \ \& \ A(e,v) \rightarrow u = v]$. Hence $\vdash_{\mathbf{HA}} \neg\exists e \forall x \exists w[T(e,x,w) \ \& \ A(x,U(w))]$, so **PA** proves the negation of an instance of CT_0.

3.4. **Axiomatization and modifications.** When formalizing the original definition, Nelson associated with each formula A of **HA** another formula $e \ \mathrm{R} \ A$ of (a conservative extension of) **HA**, then proved that every formula of the form $A \leftrightarrow \exists e(e \ \mathrm{R} \ A)$ was realizable and hence consistent with **HA**. In 1971 Troelstra used an extension ECT_0 of Church's Thesis CT_0 to determine the exact strength of number-realizability over intuitionistic arithmetic.[21] Briefly, Troelstra showed that the provable sentences of **HA** + ECT_0 are exactly those whose realizability can be established in **HA**, and that $\vdash_{\mathbf{HA}+\mathrm{ECT}_0} (E \leftrightarrow \exists x(x \ \mathrm{R} \ E))$ for every formula E.

From the intuitionistic point of view Church's Thesis is restrictive, and probably unacceptable as a general principle. Markov's Principle MP, the schema

$$\forall x(A(x) \vee \neg A(x)) \rightarrow [\neg\forall x \neg A(x) \rightarrow \exists x A(x)],$$

is also problematic. Because MP implies its own realizability, Troelstra's theorem extends over **HA** + MP to **HA** + ECT_0 + MP.[22] Kreisel [34] invented a typed version of realizability in order to show the independence of Markov's Principle.

Other modifications of realizability have been developed to prove a variety of independence and consistency results, and this process continues. The original notion gives a classically comprehensible interpretation of Heyting arithmetic but does not claim to capture intuitionistic arithmetical truth.

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[20] For details including the definition of *numeralwise expressibility* see [29].

[21] ECT_0 is the schema $\forall x[A(x) \rightarrow \exists y B(x,y)] \rightarrow \exists e \forall x[A(x) \rightarrow \exists w(T(e,x,w) \ \& \ B(x,U(w)))]$, where $A(x)$ is *almost negative* (containing no $\vee$, and no $\exists$ except immediately in front of a prime or quantifier-free formula); every formula $e \ \mathrm{R} \ A$ is of this kind.

[22] Troelstra has convincingly argued that the "Russian recursive mathematics" of the Markov school is based on this theory.

4. The intuitionistic theory of the continuum

Brouwer's main objection to classical mathematics (apart from the unrestricted use of the principle of excluded middle) was "its introduction and description of the continuum."[23] His entire work was motivated by the attempt to describe a construction of the continuum in harmony with his mathematical principles, and to develop a satisfactory mathematics on the basis of that construction.[24] To achieve this goal he created the new notions which give intuitionistic mathematics its unique character.

Our idea of the continuum, the real plane or in the one-dimensional case the real line, seems to originate from our perception of space, a primary notion which is *a priori* according to Kant. Thus, until non-Euclidean geometries appeared and the uniqueness of Euclidean space was lost, the continuum could be considered an initial concept, immediately comprehensible by intuition, not requiring analysis in terms of other more elementary concepts. After this loss, however, mathematicians began trying to define the continuum from apparently more fundamental notions, by more and more abstract methods. It was the era of Cantor's creation of the theory of sets, with the ordinal and cardinal numbers, and of the arithmetization of analysis by Weierstrass, Dedekind, Cantor and others.

Since then, in traditional classical mathematics the continuum is considered to be a collection of distinct mathematical objects, the real numbers, which are defined using Dedekind cuts or Cauchy sequences of rational numbers, or nested sequences of intervals with rational endpoints. In each case, infinite mathematical entities are treated as completed, actual. Even the (semi)intuitionists Poincaré, Borel and Lebesgue, in their treatment of the continuum, failed to maintain the constructivist standpoint. Brouwer criticized them for having "recourse to logical axioms of existence" such as the axiom of completeness, or "[contenting] themselves with an ever-denumerable and ever-unfinished" ([11] p. 5) set of numbers so that, in order to retain constructivity, they were restricted to definable and hence to only countably many objects (for example, definable sequences of rationals).[25]

Brouwer accepted as correct Cantor's argument showing that it is impossible to enumerate all the points of the continuum. His first answer to the resulting foundational problem was presented in his dissertation, where he considered the continuum as a primitive entity, "the inseparable complement of the discrete," to which it cannot be reduced:

> "However, the continuum as a whole was given to us by intuition; a construction for it, an action which would create from the mathematical intuition 'all' its points as individuals, is inconceivable and impossible" because the "mathematical intuition is unable to create other than denumerable sets of individuals." ([5] p. 45)

According to its intuitive conception, the continuum can be infinitely divided: two discrete points are connected by a "between", which is never exhausted by the insertion of new points. So it is possible "after having created a scale of ordertype

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[23] Changes in the Relation between Classical Logic and Mathematics, in [43] p. 453.

[24] As Beth says ([2] p. 422), "the central theme in intutionistic mathematics is the theory of the continuum."

[25] In addition to the problem of cardinality, the measure of a countable set of reals is zero, and that problem also concerned Brouwer from the beginning.

η [the ordertype of the rationals] to superimpose upon it a continuum as a whole" ([5], p. 45). Of course, this description of the manner in which we appear to pass from the rationals with their dense linear order to the continuum does not give a construction of the continuum, it simply indicates the relationship between the rationals and the continuum and perhaps differs little from the acceptance of an axiom of completeness, since the gaps between the rational points are covered by their (rather indefinite) "between." In any case the intuitionistic construction of the continuum remained a challenge for Brouwer; his answer to that problem was given later and stands as his most important creation.

Among various thoughts concerning the continuum, in his doctoral thesis Brouwer presented arguments for the continuum hypothesis, i.e. the conjecture that Cantor's second number class and the continuum have the same number of points: this holds primarily, in Brouwer's early opinion, because considering the continuum as a set of points we are obliged to define its points somehow, and therefore a set of the same cardinality as the second number class results.[26]

Thus, in Brouwer's thought as expressed in his dissertation, the continuum had an intuitive basis very close to its geometrical representation, like that of the semi-intuitionists. A constructive *arithmetical* (in the sense of being based in some way on some notion of number) view of the continuum was impossible, and so its mathematical treatment remained problematic.

In his lectures of his mature period, Brouwer gave to his first, essentially restrictive intervention in the problems of foundations of mathematics the name "First Act of Intuitionism."[27] With it he completely separated language and logic from mathematics and ruled that the intuition of many-oneness is the only basis for mathematical activity - as he had already stated in his dissertation. He recognized however the restricted possibilities of mathematical development (only "separable" mathematics like arithmetic and algebra survived): "Since the continuum appears to remain outside its scope, one might fear at this stage that in intuitionism would be no place for analysis." ([11], p. 7). And then he presented the "*Second Act of Intuitionism*,"[28] with which he introduced his new notions of *free choice sequences* and *species* and began the intuitionistic reconstruction of analysis.

[26]Indeed Brouwer notes that the problem has not been well posed, because "neither the second number class nor the continuum as a totality of individualized points exists mathematically" ([5], p. 83).

[27]*The First Act of Intuitionism.* The complete separation of mathematics from mathematical language, and hence from the linguistic phenomena which are described by theoretical logic, the recognition that intuitionistic mathematics is an essentially languageless activity of the mind which springs from the perception of a motion of time. This perception of time can be described as the splitting of a life moment into two distinct things, one of which gives place to the other, but is preserved in memory. If the dyad thus born is stripped of every quality, what remains is the blank form of the common substratum of all dyads. And it is this common substratum, this common form, which is the basic intuition of mathematics. ([11], p. 4)

[28]*The Second Act of Intuitionism.* The adoption of two ways of creating new mathematical entities: first in the form of infinite sequences of mathematical entities which have already been constructed, which continue more or less freely (so that, for example, infinite decimal fractions which neither have exact values nor are guaranteed to ever acquire exact values are allowed); second, in the form of mathematical species, that is properties which are supposed to hold of mathematical entities which have already been constructed, which satisfy the condition that if they hold for a mathematical entity, they also hold for all the mathematical entities which have been determined to be 'equal' to it, where the definitions of equality must satisfy the conditions of symmetry, reflexivity and transitivity. ([11], p. 8)

4.1. **Brouwerian set theory.**

4.1.1. *Choice sequences.* As we have seen, the role of infinite sequences is crucial in an arithmetical description of the continuum. Borel had remarked that the only way to obtain the continuum using only sequences of rationals without imposing the existence of real numbers by means of axioms, was to adopt sequences of arbitrary choices of objects. As a constructivist he hesitated but did not completely reject the idea:

> "...one knows that the completely arithmetical concept of the continuum requires that one admits the legitimacy of a countable infinity of successive choices. This legitimacy seems to me highly debatable, but nevertheless one should distinguish between this legitimacy and the legitimacy of an uncountable infinity....The latter concept seems to me ... entirely meaningless..." ([3] and [50] volume 2, p. 641).

Brouwer, with his Second Act of Intuitionism, accepted free choice sequences as legitimate mathematical objects.[29] As we shall see, he found a way to use these infinite, undetermined objects constructively by viewing them as having two parts: a finite, already constructed part which permits genuine constructive use of the infinite sequence under some circumstances, and an infinite, undetermined part which makes it possible to obtain the whole continuum, escaping the restrictions imposed by any sort of definability.

4.1.2. *The new notions of set.* Brouwer's new perspective on the foundations of mathematics was presented in his 1918 article "*Foundation of Set Theory, independent of the logical law of excluded middle. First part: General Set Theory.*" [8]. There, instead of Cantor's sets, two alternative notions are proposed as the basis for analysis, the *spread* (Menge)[30] and *species.*

Species: the set as property. Species are in fact closely related to sets as understood in classical mathematics. A species is a property, but a property "supposable for mathematical entities previously acquired", and is extensional with respect to the notion of equality (in general an equivalence relation) between the mathematical objects in question. The verification that a mathematical object has a property (hence "belongs to the species") requires a construction; the species therefore may be thought of as a construction *within* a construction. Two species are the same "if to every element of each an equal element of the other can be correlated." With this definition of species, the problems of impredicativity and self reference are avoided: "a species can be an element of another species, but never an element of itself!" as Brouwer himself asserts ([11], p. 8).

Spread: the set as law. But apart from the natural numbers, which mathematical entities can be elements of species? The generation of mathematical objects results from mathematical activity typically involving the radically new concept of spread, where free choice sequences are allowed. Each element of a spread is constructed in

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[29]This acceptance reflected his opinion, inspired by Kant, that mathematical constructions are internal, free creations of the mind of the *creating subject*, a view which influenced his methodology.

[30]Brouwer initially used the term Menge, which had been preempted by Cantor, but later preferred the term spread. In the Greek version of this article we translated "spread" using an ancient Greek word for loom, following the example of Wim Veldman (possibly the unique modern mathematician who works precisely in the spirit of Brouwer).

stages: at each stage we choose, in accordance with certain rules, one more piece of the element, thus obtaining a better approximation of it. Specifically:

A spread is created i) by successive choices of natural numbers; each choice depends on the preceding ones, and is made either freely or in accordance with certain restrictions, and ii) by assigning after each choice an object from a particular preexisting denumerable set. The restrictions in i) and the correlations in ii) are given by the *spreadlaw*, in reality a pair of laws, the *choice law* (which also determines whether the process will terminate or be continued, and in the second case there must be at least one permitted choice for the next step)[31] and the *correlation law*. Attempting to describe the nature of the 'laws', Brouwer notes: "...we can say that a spread law yields an *instruction*[32] according to which ..." ([11], p. 15).

The *elements* of the spread are the sequences (infinite or finite) of the objects correlated to each sequence of choices. So an element may remain forever incomplete, always becoming. The spread thus created is not considered to be the totality of its elements, but is identified with the law of its creation. The description of the notion of spread is complicated and long, but Brouwer said he could not avoid this complexity. Over time he made various modifications to the details of the definition, concerning for example the nature of the restrictions imposed.

Despite the fact that every choice and subsequent assignment is made effectively and so it is decidable whether a given finite sequences of choices or of objects to be assigned is permitted, the same is not true for the infinite sequences: a free choice sequence is constructed in steps; if at some step the choice law determines that the finite part which has already been created no longer satisfies the restrictions, then indeed we know that the sequence does not 'belong' to the spread. But as long as it satisfies the restrictions, we can never be sure that it will not be rejected at some later step, and that is the price of freedom in the creation of the sequence.

Spreads are more basic objects than species: the spread generates its elements, while the species has elements already constructed, and what is needed for a mathematical entity to belong to a species is the intuitionistic justification that it has the property which defines the species.

4.1.3. *Spreads as trees with a topology.* We can think of spreads as trees, whose nodes are the finite sequences of correlated objects corresponding to the successive choices, and all of whose branches are infinite (a branch is a sequence each of whose finite initial segments belongs to the tree). Brouwer himself used this image in describing spreads.

In a very natural way a topology can be defined on a spread: considering the infinite branches (using the tree image) as points and taking as basis for the topology the set $\mathcal{N}_u$, where $\mathcal{N}_u$ is the set of the infinite branches with initial segment u.

4.1.4. *Examples.* Now we give two most important examples of spreads.

1. The *universal spread*, whose choice law permits the choice of any natural number whatsoever at each step, and whose correlation law assigns trivially after each choice the number just chosen. This spread gives us all the sequences of natural numbers, so the species of its elements is uncountable. Equipped with the initial segment topology, it becomes the familiar Baire space, and so it is Borel-isomorphic

[31] This means that we can assume that only infinite sequences are produced by the spread law.
[32] Italics ours.

(isomorphic as a measure space) to the set $\mathcal{R}$ of the traditional real numbers; it is (topologically) homeomorphic with the irrational numbers. This most general and free process associated with the universal spread captures the essence of the creation of the intuitionistic continuum, and on this basis, by placing a few minor restrictions, we can define the real numbers in the context of intuitionism.

2. A finitely branching spread is called a *fan*. Fans, which are compact in the above topology, play a special role in the development of a theory of analysis because of their more determinate character. One fan of particular significance is the binary fan, consisting of all infinite sequences of 0s and 1s, which with the initial segment topology is the well-known Cantor space.

The intuitionistic continum and the real numbers are the next examples, to which we now turn.

4.2. **The intuitionistic continuum.**

4.2.1. *The spread of the points of the continuum and the real numbers.* Let us consider[33] the species of the binary fractions $a/2^n$, where a is an integer and n a natural number, with the usual ordering. In this species, we consider the closed intervals of the form $I_{m,n} = [m/2^n, (m+2)/2^n]$ and an enumeration of the pairs of natural numbers $p_1, p_2, ...$ with $p_i =< r_i, s_i >$, and the following spread: at the first step the choice of any natural number n is permitted and the interval I_{r_n,s_n} is assigned; if $a_1, ..., a_n$ have been chosen and the interval $I_{r,s}$ has been assigned, then it is permissible to choose the number k only if I_{r_k,s_k} is properly contained in $I_{r,s}$, and if k is chosen then the interval I_{r_k,s_k} is assigned. The elements of this spread, these infinitely proceeding sequences of closed intervals whose lengths converge to 0, are *the points of the continuum.*[34]

An equivalence relation is now defined for these points: two points p and q *coincide* if every interval of p intersects every interval of q. Each equivalence class is a species called a *point core* and this is what a real number is defined to be. The species of all these point cores is *the intuitionistic continuum.* The equality between reals is obtained from the coincidence relation: if r and s are reals and p, q representatives of r, s correspondingly, then $r = s$ if and only if p and q coincide.

Brouwer gave many analogous constructions of the continuum, using for example sequences of rationals or binary fractions with various rates of convergence. Heyting also gave some constructions of this kind, using the characteristic term *real number generators* for the corresponding point cores. This description of the real numbers solves the problems of cardinality and measure from an intuitionistic standpoint. For example, the rationals are naturally embedded into the reals defined as above, since every rational belongs to a sequence of intervals $I_{m,n}$ with lengths tending to 0. But the differences from the real numbers of classical mathematics remain large, as will appear in what follows.

4.2.2. *Undecidability of equality.* In his 1930 paper titled "*The structure of the continuum*" [10] Brouwer examined basic properties of the continuum, comparing the classical continuum with the one resulting from his own views. His first conclusion

[33] From the footnote of Parsons in [9].

[34] We can compare this process with the *nested interval principle*, a completeness axiom which assures that the intersection of every sequence of nonempty closed nested intervals whose lengths converge to 0 is nonempty.

was that equality between two real numbers *is undecidable* in the case of the intuitionistic continuum. We can see this as follows: Let $A(n)$ be a property which is decidable for each n, for example the Goldbach conjecture (every even number greater than 3 is the sum of two primes), so that $A(n)$ is the property that $2n+4$ is the sum of two primes. Let

$$\alpha(n) = \begin{cases} 1/2^n, & \text{if } \forall k \leq n\ A(k), \\ 1/2^k, & \text{if } \neg A(k)\ \&\ k \leq n\ \&\ \forall m < k\ A(m). \end{cases}$$

The sequence of rationals so defined is evidently convergent, hence determines a real number r. We see that $r = 0$ if and only if $A(n)$ holds for every n (otherwise $r = 1/2^k$ for the least k for which $A(k)$ fails). So we cannot decide if r coincides with 0, as long as we do not know the answer to Goldbach's conjecture.

4.2.3. *Weak counterexamples.* Counterexamples of the type just considered are characteristic of Brouwer's argumentation, and are known as *weak counterexamples.* Each one uses a problem which has not yet been solved in order to deny a classically valid statement. It is weak because it depends crucially on a *specific* unsolved problem, to which a solution may someday be found; for instance, many of Brouwer's weak counterexamples involved Fermat's Last Theorem, which was then undecided but has now been proved.

But Brouwer believed that there would always be unsolved mathematical problems. A weak counterexample can be considered as a construction which will convert any unsolved problem into a refutation of a classical theorem. Implicit in the use of weak counterexamples is Brouwer's continuity principle, to be explained below.

4.2.4. *The continuum cannot be ordered.* A second result was the impossibility of finding a linear order of the intuitionistic continuum. The rationals are obviously totally ordered, because the comparison of rationals reduces to the comparison of natural numbers. For the reals, some can certainly be compared and ordered among themselves; in the characterization by nested intervals for example, if at some step of the construction of two points of the continuum p and q an interval of p lies entirely to the left of an interval of q, then, for the corresponding reals r and s we have $r < s$. Thus the *natural order* of the reals, as Brouwer called it, is derived. But again by appealing to weak counterexamples, he showed that this order cannot be total. Then he defined a refinement of the natural order, the *virtual order*

$$r \prec s \equiv \neg r > s\ \&\ r \neq s,$$

which also fails to be total. It is worth noting that, although from the intuitionistic standpoint not all real numbers are comparable, the natural ordering can be proved to satisfy certain properties; the most useful substitute for comparability is the property $r < s \rightarrow r < t \vee t < s$. The lack of order is not a "technical" problem, but rather a consequence of the fact that most real numbers are incompletely determined objects.

4.2.5. *The unit continuum.* Because two real numbers r and s are not always comparable with respect to the natural ordering, the closed interval $[r, s]$ is defined to be the species of all reals x for which it is impossible that $x > r$ and $x > s$, and also impossible that $x < r$ and $x < s$. If $r < s$ then $[r, s]$ is just the species of x such that neither $x < r$ nor $x > s$. The proof of the uniform continuity theorem, discussed below, depends on the fact that any closed interval $[r, s]$ is the continuous image of a fan F of real number generators, each of which coincides with a point of $[r, s]$. By the *unit continuum* Brouwer meant a fan coordinated in this way with $[0, 1]$ (as in [11] p. 35; cf. [50] and [35]).

4.3. **The basic theorems and implicit principles.**

4.3.1. *Discontinuous functions do not exist.* The undecidability of equality has consequences which a traditional mathematician would find hard to accept. Let[35] f be the function from the reals to the reals with

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

which is classically defined everywhere and discontinuous at 0. Intuitionistically however, f cannot be a total function: for the real number r of the weak counterexample we gave, as for any other real, it follows by continuity (in the usual sense) that we can calculate any approximation to $f(r)$, and from the presumed totality of the function we can decide whether $f(r) < 1$ or $f(r) > 0$, and so decide whether $\neg r \neq 0$ or $r \neq 0$, or equivalently[36] whether $r = 0$ or $r \neq 0$, which is impossible; thus we arrive at the conclusion that f cannot be a total function.

Another impressive result obtained by similar arguments is that the Intermediate Value Theorem fails intuitionistically.

4.3.2. *The uniform continuity theorem.* In contrast with these negative results, Brouwer reached a very strong conclusion, similar to the classical theorem that functions continuous on a compact space are uniformly continuous, except that in the intuitionistic case only continuous functions exist:

Uniform continuity theorem. Every total function defined on the unit continuum is uniformly continuous.

In fact Brouwer probably arrived at this result guided by his intuition rather than by clear mathematical arguments. Hermann Weyl said the following in one of his lectures, defending Brouwer's continuum: "It is clear that no one can explain the meaning of 'continuous function on a bounded interval' without including 'uniform continuity' and boundedness in the definition. Above all, there can be no function on a continuum other than continuous functions." (citeVS1990, p. 379). Brouwer read and emphatically agreed with these remarks. For many years he tried to find a satisfactory proof for this theorem. In his 1927 article "*On the domains of definition of functions*" [9] the most complete presentation of his arguments is given. The value of this attempt consists mainly in the fact that in his proof, especially in its final form, the two characteristic principles of intuitionistic analysis (which we will discuss below) are used clearly and thus brought to light, though not explicitly as principles but only as natural manifestations of the intitionistic viewpoint.

4.3.3. *The Fan Theorem.* The uniform continuity theorem was obtained as a consequence of the fan theorem, as Brouwer called it.

The Fan Theorem. If to each element α of a fan a number b_α is associated, then a natural number z can be found so that b_α is determined completely by the first z values of α.

[35] The example is from [50], volume 1, p. 14.

[36] The equivalence of $\neg r \neq 0$ with $r = 0$ needs a little argument. The equality $r = 0$ is of the form $\forall n\ A(n)$, and the $A(n)$ is decidable hence stable under double negation, while the implication $\neg\neg\forall n A(n) \rightarrow \forall n \neg\neg A(n)$ holds intuitionistically.

This version of the theorem fails classically; its proof uses the (classically correct) method of backwards induction but also the (classically false) intuitionistic continuity principle[37]. It is the form which Brouwer preferred.

A different version of the theorem is provable just using the principle of backwards induction and is classically correct. In this version, the theorem says that if, for the given fan there is a decidable set B of nodes such that each branch α meets B (such a B is called a *bar*), then there is a number z such that each branch meets the bar at a node of length at most z. We mention this alternative because it is classically equivalent to König's Lemma (every infinite, finitely branching tree has an infinite branch).

As van Dalen notes in [12], "it is an interesting historical curiosity that the fan theorem preceded its much better known contrapositive: König's infinitary lemma [König 1926] ... The infinitary lemma is not constructively valid. An interesting observation is that this contradicts the popular impression that constructive proofs and theorems are always later improvements of classical theorems."

4.3.4. *The Bar Theorem.* In order to prove the fan theorem Brouwer first gave a complicated proof, more or less metamathematical in character, of what he later called the *bar theorem*, in which he examined the structure of possible proofs of sentences of certain forms. Later however, as appears in a footnote to his 1927 article [9], he realized that what he uses is in fact a kind of induction principle. A statement of this principle follows:

Bar Theorem. Let the universal spread contain a decidable bar A and X be a set of nodes satisfying (i) $A \subset X$ and (ii) if all the immediate successors of a node n belong to X, then n belongs to X; then it follows that the empty sequence (the root of the universal spread) belongs to X.

We remark that this principle makes it possible to exploit properties possessed by all sequences of natural numbers, despite their incomplete character, in the case where these properties can be verified at some finite stage of the generating process of the sequence. But in any case the assumption of the existence of a bar on the universal spread is not at all trivial; to see this, we need only observe that the set of nodes where each branch meets the bar for the first time may be as "long" as any infinite countable ordinal number. While this principle seems to be closely connected with intuitionistic mathematics, it also happens to be classically correct.

4.3.5. *The continuity principle.* When Brouwer proved that the universal spread generates uncountably many elements, he replaced Cantor's diagonal argument by a new argument of exceptional simplicity, as follows: suppose we have a function from the universal spread to the natural numbers, and let b be the value this function takes on the sequence α. Constructively, this b must be determined by the first y values of α, for some y. But then the function must also take the value b on a sequence β having the same first y terms as α, although $\beta(y+1) \neq \alpha(y+1)$. Hence it is impossible to map the universal spread to the natural numbers in a one-to-one way, and so the elements of the universal spread cannot be enumerated.

The substance of this argument is *the continuity principle*, which played a crucial role in the proofs of several theorems considered above. According to this principle,

[37]The principles mentioned here will be defined below.

in order for a function to be defined on a spread, the value it takes on each sequence-argument must be determined entirely by an initial segment of the sequence. This is how the conflict between the incompletely determined nature of choice sequences, and the constructive character of a function defined over them, was reconciled by Brouwer. So the principle postulates that every total function is continuous. It can also be generalized to functions with sequence values. It is the only genuinely intuitionistic principle of Brouwer's mathematics, it contradicts classical analysis, and thus the mathematics based on it are distinct from, in fact inconsistent with, classical mathematics.

4.3.6. *Other mathematical principles.* It is certain that Brouwer saw the general axiom of choice as eminently nonconstructive. However various arithmetical forms of it result by correct reasoning from the manner in which existential and universal statements are understood intuitionistically. We will discuss these in the context of a formal system which was proposed for Brouwer's analysis.

4.4. **$\mathcal{FIM}$: a formal system for intuitionistic analysis.** The axiomatization of intuitionistic mathematics by Heyting in his classic work [19, 20, 21] included the theory of sets, but in a form which, as Kleene observes, did not facilitate its comparison with classical mathematics. Kleene and Vesley's book "The Foundations of Intuitionistic Mathematics" [30], which was published in 1965, was the result of many years' research. It presented a formal system $\mathcal{FIM}$ for a part of intuitionistic mathematics including Brouwer's set theory restricted to definable properties of numbers and sequences, sometimes referred to as intuitionistic analysis.[38] The language he uses is identical with that of a formal system for classical analysis. However, in contrast to intuitionistic propositional and predicate logic and Heyting arithmetic, intuitionistic analysis *is not* a subsystem of the classical theory, but a divergent theory of analysis, as would be expected from the discussion above. However, $\mathcal{FIM}$ contains a subsystem, the *basic system* $\mathcal{B}$, consisting of the common part of classical and intuitionistic analysis. When $\mathrm{X}10^c$ (the principle of double negation) is added to $\mathcal{B}$ the result is a corresponding system for classical analysis, while the intuitionistic system $\mathcal{FIM}$ comes from $\mathcal{B}$ by adding one axiom expressing the classically false continuity principle.

The language of $\mathcal{FIM}$ is a two-sorted first order language with equality, suitable for a formal theory of natural numbers and choice sequences (functions from ω to ω); we shall see that it can also express continuous functionals. There are two types of individual variables, arithmetical variables a,b,c,..., x,y,a, ... and function variables $\alpha, \beta, \gamma, \ldots$, and finitely many function constants $\mathrm{f}_0, \ldots, \mathrm{f}_{24}$, expressing zero, successor, addition and multiplication, as well as other primitive recursive functions such as predecessor, remainder and quotient; each f_i expresses a function of k_i number and l_i function arguments, respectively. This choice is not unique, it was intended by Kleene to simplify the development of the theory.[39] In addition, the symbols of the language include Church's λ.

[38] Kleene himself referred to this part as "two-sorted intuitionistic number theory."

[39] The complete catalog of function constants can be found in [30]. In the spirit of Brouwer it is open, allowing additions as required by the development of the theory; thus, in [31] two more function constants are proposed.

In addition to the *terms*, which are the formal expressions of $\mathcal{FIM}$ for the natural numbers, in this system there are formal expressions, called *functors*, for (total) functions from ω to ω. The two notions are defined by simultaneous induction:

(i) The number variables are *terms*.
(ii) The function variables are *functors*.
(iii) For every $i = 0, \ldots, 24$, if $k_i = 1$ and $l_i = 0$, then f_i is a *functor*.
(iv) For every $i = 0, \ldots, 24$, if $\mathrm{t}_1, \ldots, \mathrm{t}_{k_i}$ are *terms* and $\mathrm{u}_1, \ldots, \mathrm{u}_{l_i}$ are *functors*, then $\mathrm{f}_i(\mathrm{t}_1, \ldots, \mathrm{t}_{k_i}, \mathrm{u}_1, \ldots, \mathrm{u}_{l_i})$ is a *term*.
(v) If u is a *functor* and t is a *term*, then $(\mathrm{u})(\mathrm{t})$ is a *term*.
(vi) If x is a number variable and t is a *term*, then $\lambda\mathrm{x}\,\mathrm{t}$ is a *functor*.
(vii) An expression is a *term* or *functor* if and only if that follows from (i)– (vi).

The atomic or prime formulas are expressions of the form $\mathrm{s} = \mathrm{t}$ where s and t are terms. Equality between functors is not prime, but is defined by

$$\mathrm{u} = \mathrm{v} \;\equiv\; \forall\mathrm{x}\,(\mathrm{u}(\mathrm{x}) = \mathrm{v}(\mathrm{x})),$$

where x does not appear free in u, v. The *formulas* are formed as usual, with the difference that here the quantifiers may apply to function variables, so that $\forall\alpha\mathrm{A}$, $\exists\alpha\mathrm{A}$ are formulas.

The free and bound occurrences of variables are defined as usual, except that Church's λ symbol also binds number variables.

The axiom schemas and rules of inference of the system include first of all X1 - X12 and R1 - R3 (for the language of $\mathcal{FIM}$) as well as the following, where u is a functor free for α in $\mathrm{A}(\alpha)$ and α does not occur free in C:

X13. $\forall\alpha\ \mathrm{A}(\alpha) \to \mathrm{A}(\mathrm{u})$.
X14. $\mathrm{A}(\mathrm{u}) \to \exists\alpha A(\alpha)$.
R4. From $\mathrm{C} \to \mathrm{A}(\alpha)$, conclude $\mathrm{C} \to \forall\alpha\ \mathrm{A}(\alpha)$.
R5. From $\mathrm{A}(\alpha) \to \mathrm{C}$, conclude $\exists\alpha\ \mathrm{A}(\alpha) \to \mathrm{C}$.

The equality axioms XE1 - XE3 and the arithmetical axioms XInd and XN1 - XN7 are also included, where of course x, y, z are number variables. The axioms concerning functors are first of all the definitions of the function constants $\mathrm{f}_0, \ldots, \mathrm{f}_{24}$, which are formulas expressing the corresponding explicit or recursive definitions, as well as:

XF1. $(\lambda\mathrm{x}\,\mathrm{r}(\mathrm{x}))(\mathrm{t}) = \mathrm{r}(\mathrm{t})$.
XF2. $\mathrm{a} = \mathrm{b} \to \alpha(\mathrm{a}) = \alpha(\mathrm{b})$.
XF3. $\forall\mathrm{x}\,\exists\alpha\,\mathrm{A}(\mathrm{x},\alpha) \to \exists\alpha\,\forall\mathrm{x}\,\mathrm{A}(\mathrm{x},\lambda\mathrm{y}\,\alpha(2^{\mathrm{x}} \cdot 3^{\mathrm{y}}))$,

where $\mathrm{r}(\mathrm{x})$, t are terms, x is a number variable, t is free for x in $\mathrm{r}(\mathrm{x})$, and $\mathrm{A}(\mathrm{x},\alpha)$ is a formula in which x is free for α.

XF1 is the β-reduction axiom of the λ-calculus; XF3 is a principle of countable choice which is acceptable according to the intuitionistic interpretation of the quantifiers.

For the description and treatment of choice sequences in $\mathcal{FIM}$ we use *sequence numbers*, which code initial segments of sequences of natural numbers as follows: the first x values of a sequence α are given by $\overline{\alpha}(x)$, where

$$\overline{\alpha}(0) = 1 \text{ and } \overline{\alpha}(x+1) = p_0^{\alpha(0)+1} \cdot \ldots \cdot p_x^{\alpha(x)+1},$$

where $p_0, p_1, \ldots$ are the prime numbers in their natural order. This coding, as well as the formal predicate Seq(a) expressing that a codes a finite sequence, can be defined in the formal system. So can the operation $\mathrm{a} * \mathrm{b} = \mathrm{a} \cdot \prod_{\mathrm{i}<\mathrm{lh(b)}} \mathrm{p}_{\mathrm{lh(a)+i}}^{(\mathrm{b})_\mathrm{i}}$ [40] of concatenation. With these tools it becomes possible to formulate in $\mathcal{FIM}$ the characteristic principles of Brouwer's mathematics concerning choice sequences.

The mathematical principle which Brouwer tried to convey by the bar theorem is introduced in $\mathcal{FIM}$ by the Axiom of Backwards Induction (Bar Induction):

$$\begin{aligned}\mathcal{BI}: \quad & \forall \mathrm{a}\,[\,\mathrm{Seq(a)} \rightarrow \mathrm{R(a)} \vee \neg \mathrm{R(a)}\,] \ \& \ \forall \alpha\, \exists \mathrm{x}\, \mathrm{R}(\overline{\alpha}(\mathrm{x})) \ \& \\ & \forall \mathrm{a}\,[\,\mathrm{Seq(a)} \ \& \ \mathrm{R(a)} \rightarrow \mathrm{A(a)}\,] \ \& \\ & \forall \mathrm{a}\,[\,\mathrm{Seq(a)} \ \& \ \forall \mathrm{s}\, \mathrm{A}(\mathrm{a} * 2^{\mathrm{s}+1}) \rightarrow \mathrm{A(a)}\,] \\ & \rightarrow \mathrm{A}(1).\end{aligned}$$

About Kleene's choice of this schema we can say the following:
The properties of choice sequences which are interesting from the intuitionistic viewpoint are those which are verifiable on the basis of some initial part of each choice sequence, that is they are of the form $\exists x R(\overline{\alpha}(x))$, where $R(a)$ is an arithmetical predicate, decidable at least for sequence numbers. $\mathcal{BI}$ expresses the following induction principle for spreads (trees) on which there is a (decidable) bar: if (i) $A(a)$ is a property which is possessed by sequence numbers having the property $R(a)$ and (ii) $A(a)$ is transmitted[41] recursively, in the sense that for every sequence number a, if $A(a * 2^{s+1})$ holds for every s, then also $A(a)$ holds, then we conclude that $A(1)$ holds, where 1 is the code of the empty sequence.
$R(a)$ plays the role of the bar; this form of the axiom only takes care of the case of the universal spread, but the notion of spread is definable in $\mathcal{FIM}$ and the general form is provable from this.

The important Fan Theorem, formally stated, is proved in $\mathcal{FIM}$ using $\mathcal{BI}$.

$\mathcal{BI}$ is provable classically (and without the restriction of decidability of $R(a)$, which is meaningless in this case); together with the preceding axioms and rules, it completes the basic system $\mathcal{B}$.

The next and final axiom expresses the classically false continuity principle. For the formulation of this principle in $\mathcal{FIM}$ Kleene points out that Brouwer himself, speaking about the assignment of a value b to a function α, used the expression "*the algorithm of the correlation law*". Taking this expresssion seriously and arguing as follows:

first, the algorithm for computing the value b must (i) decide, for each initial segment $\alpha(0), \ldots, \alpha(y-1)$ of a choice sequence α, whether these values suffice for the computation of b; (ii) if the answer is "yes," compute b, and

second, this information must be given by a function τ, which acts on sequence numbers $\overline{\alpha}(y)$ and produces 0 as long as the algorithm does not answer "yes," while if y is the least number for which the answer is "yes" then $\tau(\overline{\alpha}(y)) = b+1$ (and we may assume without loss of generality that this is the only y for which $\tau(\overline{\alpha}(y)) > 0$),

Kleene proposed the following formal statement of the continuity principle (Brouwer's principle):

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[40] Here lh(a) expresses the length of the sequence coded by a, and $(\mathrm{a})_\mathrm{i}$ its i^{th} projection.

[41] In the direction from the bar *toward* the root of the tree, hence the name *backwards* induction.

$$\mathcal{BP}: \quad \forall\alpha\,\exists\beta\,A(\alpha,\beta) \to \exists\tau\forall\alpha\,[\forall t\,\exists!y\,\tau(2^{t+1} * \overline{\alpha}(y)) > 0 \,\&$$
$$\forall\beta\,[\forall t\,\exists y\,\tau(2^{t+1} * \overline{\alpha}(y)) = \beta(t) + 1 \to A(\alpha,\beta)\,]\,].$$

This principle is false for classical mathematics. For example, in [30] it is used to prove

$$\vdash \neg\forall\alpha(\forall x\,\alpha(x) = 0 \vee \neg\forall x\,\alpha(x) = 0)$$

(from which it follows that equality between choice sequences must be undecidable), and the negation of the universal closure of the least number principle.

In addition, the third chapter of [30] contains R. E. Vesley's presentation of the intuitionistic theory of the continuum in the context of the formal system $\mathcal{FIM}$. Real numbers are defined by formalizating the notion of real number generator and the uniform continuity theorem is proved. In the fourth and last chapter, Kleene treats formally the question of the order of the reals. Thus an important part of Brouwer's mathematics is faithfully represented in this formal system.

4.5. **Function realizability interpretation of the system $\mathcal{FIM}$.** In the same book, by adapting the basic idea of number realizability, Kleene gives an interpretation of intuitionistic analysis which guarantees the consistency of the formal system. Here we have propositions of the form $\exists\beta\,A(\beta)$, where β is a function variable; to interpret and justify such a proposition requires a function β and a verification that the interpretation of $A(\beta)$ is correct. Now the suitable candidates for realizing objects are not natural numbers, but *number-theoretic functions of one (number) variable.* The basic mechanism Kleene used, in analogy with the coding of recursive functions by natural numbers, is the coding of (continuous) functionals $F : \omega^\omega \to \omega^\omega$ by means of functions $\tau : \omega \to \omega$. Thus he defined the partial recursive function $\{\tau\}[\alpha]$ of two function variables τ and α, which computes the functional coded by τ at the argument α.

The definition of function realizability is recursive and is motivated by the previous discussion.[42] The concept to be defined is

$$\epsilon\; r_\Psi\; \mathrm{E}: \quad \epsilon \text{ realizes E with respect to } \Psi,$$

where $\epsilon : \omega \to \omega$, E is a formula, Ψ is a list of variables containing all those occurring free in E, and Ψ is an assignment of numbers and (one-place) number theoretic functions to the variables of Ψ.[43]

1. $\epsilon\, r_\Psi$ P, where P is prime $\Longleftrightarrow_{\mathtt{df}}$ P is true for Ψ, that is, P holds for the values Ψ assigns to the variables of Ψ. $\qquad \mathcal{R}_P$
2. $\epsilon\, r_\Psi$ A & B $\Longleftrightarrow_{\mathtt{df}}$ $(\epsilon)_0\, r_\Psi$ A and $(\epsilon)_1\, r_\Psi$ B. $\qquad \&\,\mathcal{R}$
3. $\epsilon\, r_\Psi$ A $\vee$ B $\Longleftrightarrow_{\mathtt{df}}$ if $(\epsilon(0))_0 = 0$ then $(\epsilon)_1\, r_\Psi$ A and if $(\epsilon(0))_0 \neq 0$ then $(\epsilon)_1\, r_\Psi$ B. $\qquad \vee\,\mathcal{R}$
4. $\epsilon\, r_\Psi$ A $\to$ B $\Longleftrightarrow_{\mathtt{df}}$ for every α $(: \omega \to \omega)$, if $\alpha\, r_\Psi$ A then $\{\epsilon\}[\alpha]\!\downarrow$ and $\{\epsilon\}[\alpha]\, r_\Psi$ B. $\qquad \to\!\mathcal{R}$
5. $\epsilon\, r_\Psi$ $\neg$A $\Longleftrightarrow_{\mathtt{df}}$ for every α, it is not the case that $\alpha\, r_\Psi$ A (equivalently, if and only if $\epsilon\, r_\Psi$ (A $\to 1 = 0$), because of 4 and 1: there is no α such that $\alpha\, r_\Psi\; 1{=}0$, where $1{=}0$ is a false $\qquad \neg\mathcal{R}$

[42] It also uses the definitions $(\epsilon)_i = \lambda t\; (\epsilon(t))_i$ for i = 0,1, and $\{\epsilon\}[x] = \{\epsilon\}[\lambda t\; x]$.

[43] We read $\downarrow$ as "is defined".

prime formula).

6. $\epsilon\ r_\Psi\ \forall \mathrm{x}\, \mathrm{A} \quad \Longleftrightarrow_{\mathtt{df}}$ for every $x\ (\in \omega)$, $\{\epsilon\}[x]\downarrow$ and $\{\epsilon\}[x]\ r_{\Psi,\,x}\ \mathrm{A}$, where x is the value of x. $\quad \forall N\mathcal{R}$

7. $\epsilon\ r_\Psi\ \exists \mathrm{x}\, \mathrm{A} \quad \Longleftrightarrow_{\mathtt{df}}$ $(\epsilon)_1\ r_{\Psi,\,(\epsilon(0))_0}\ \mathrm{A}$, where $(\epsilon(0))_0$ is the value of x. $\quad \exists N\mathcal{R}$

8. $\epsilon\ r_\Psi\ \forall \alpha\, \mathrm{A} \quad \Longleftrightarrow_{\mathtt{df}}$ for every α, $\{\epsilon\}[\alpha]\downarrow$ and $\{\epsilon\}[\alpha]\ r_{\Psi,\,\alpha}\ \mathrm{A}$, where α is the value of α. $\quad \forall F\mathcal{R}$

9. $\epsilon\ r_\Psi\ \exists \alpha\, \mathrm{A} \quad \Longleftrightarrow_{\mathtt{df}}$ $\{(\epsilon)_0\}\downarrow$ and $(\epsilon)_1\ r_{\Psi,\{(\epsilon)_0\}}\ \mathrm{A}$, where $\{(\epsilon)_0\}$ is the value of α. $\quad \exists F\mathcal{R}$

A *closed* formula E is *realizable*, if it is realized by some *general recursive function* $\epsilon : \omega \to \omega$. An *open* formula is *realizable*, if its universal closure is.

The appropriate soundness theorem holds:
Theorem. (Kleene). Let Γ be a (finite) list of formulas and E a formula. Then, if $\Gamma \vdash_{\mathcal{FIM}} \mathrm{E}$ and the formulas of Γ are realizable, it follows that E is realizable.

Corollary. The formal system $\mathcal{FIM}$ is consistent.

Completing the presentation of function realizability, Kleene made the conjecture that, with the formalization of this notion and of the (model theoretic) proof of the consistency of $\mathcal{FIM}$, a metamathematical proof of the relative consistency of $\mathcal{FIM}$ with respect to the basic system $\mathcal{B}$ would be obtained. In 1969, in his monograph "Formalized recursive functionals and formalized realizability" [**?**], he presented a detailed formalization of the theory of recursive functions of type 2 and the corresponding formal notion of function realizability[44] as well as a variation which made it possible to prove his conjecture, and also to prove the disjunction and existence properties and a form of Church's thesis, for $\mathcal{FIM}$.

Corollary. (i) If $\vdash_{\mathcal{FIM}} \mathrm{A} \vee \mathrm{B}$ where $\mathrm{A} \vee \mathrm{B}$ is a closed formula, then one of the following holds: $\vdash_{\mathcal{FIM}} \mathrm{A}$ or $\vdash_{\mathcal{FIM}} \mathrm{B}$.

(ii) If $\vdash_{\mathcal{FIM}} \exists \mathrm{x}\, \mathrm{A}(\mathrm{x})$ where $\exists \mathrm{x}\, \mathrm{A}(\mathrm{x})$ is a closed formula, then, for some natural number x, $\vdash_{\mathcal{FIM}} \mathrm{A}(\mathbf{x})$, where $\mathbf{x}$ is the numeral for x.

(iii) If $\vdash_{\mathcal{FIM}} \exists \alpha\, \mathrm{A}(\alpha)$ where $\exists \alpha\, \mathrm{A}(\alpha)$ is a closed formula, then $\vdash_{\mathcal{FIM}} \exists \alpha_{\mathrm{GR}(\alpha)} \mathrm{A}(\alpha)$, where $\mathrm{GR}(\alpha)$ is a formula (with only α free) expressing that the function represented by α is general recursive.

In 1973 Troelstra characterized function realizability using the schema GC_1, a generalization of the continuity principle $\mathcal{BP}$.[45]

4.6. **Relativized and modified realizabilities.** Kleene relativized his original function-realizability interpretation in various ways. If Φ is a class of number-theoretic functions closed under "recursive in" then for $\Theta, \Psi \subseteq \Phi$ and $\varepsilon \in \Phi$ the notion "ε $\Phi/$*realizes*-Ψ E" is defined like "ε realizes-Ψ E" except that in Clauses 4, 5 and 8 the α is restricted to Φ.

Then E is $\Phi/$*realizable*$/\Theta$ if for some ε recursive in Θ: ε $\Phi/$realizes $\forall$E. The soundness theorem extends to the relativized notions. Using his example of a binary fan all of whose recursive branches (but not all of whose branches) are finite, and

[44] As in the arithmetical case, what is involved is a translation, by which to each formula E there corresponds a formula ε r E.

[45] GC_1 is the schema $\forall\alpha(\mathrm{A}(\alpha) \to \exists\beta\mathrm{B}(\alpha,\beta)) \to \exists\sigma\forall\alpha(\mathrm{A}(\alpha) \to \exists\beta(\{\sigma\}[\alpha] = \beta\ \&\ \mathrm{B}(\alpha,\beta)))$, where A is an almost negative formula.

taking $\Phi = \Theta$ to be the class of all general recursive functions, Kleene showed that Brouwer could not have proved his "Bar Theorem" without using bar induction in the proof.

Theorem. (Kleene) The axiom schema of bar induction is independent of the other axioms of intuitionistic analysis. If the "fan theorem" replaces the bar induction schema, then it too is independent of the other axioms.[46]

Next he took $\Phi = \Theta$ to be the class Ξ of all arithmetical functions, which is (classically) closed under general recursiveness and the jump operation, and proved (classically) that the fan theorem and all axioms of the intuitionistic system *except* bar induction are Ξ/*realizable*/Ξ . (Later, Howard and Kreisel [26] proved that bar induction actually changes intuitionistic arithmetic; Troelstra [46] proved that the fan theorem does not.)

Kreisel [34] first suggested a different kind of modification of realizability (later adapted by Kleene) in order to prove Markov's Principle independent of the intuitionistic axioms. Kreisel and Kleene used explicit types, but the same effect is obtained using implicit types via a notion of *agreement.* Van Oosten [41] explains the main idea: "Each formula gets *two* sets of realizers, the *actual* realizers being a subset of the *potential* ones."

For example, ε *agrees with* $\mathrm{A} \to \mathrm{B}$ if, whenever σ *agrees with* A, $\{\varepsilon\}[\sigma]$ is defined and *agrees with* B. If F is a collection of functions closed under "general recursive in" and the free variables of $\mathrm{A} \to \mathrm{B}$ are interpreted by numbers and functions from F, then ε ${}^{\mathsf{F}}$*realizes* $\mathrm{A} \to \mathrm{B}$ under this interpretation of the variables if $\varepsilon \in \mathsf{F}$ and ε agrees with $\mathrm{A} \to \mathrm{B}$ and for every $\sigma \in \mathsf{F}$: if σ ${}^{\mathsf{F}}$*realizes* A under the interpretation then $\{\varepsilon\}[\sigma]$ ${}^{\mathsf{F}}$*realizes* B under the interpretation.

The proof that bar induction is ${}^{\mathsf{F}}$realizable is a little more complicated than for realizability. The critical observation is that if σ ${}^{\mathsf{F}}$realizes $\forall\alpha\exists!\mathrm{x}\mathrm{R}(\overline{\alpha}(\mathrm{x}))$, then $\{\sigma\}[\varphi]$ is defined for *every* φ by agreement, and $(\{\sigma\}[\varphi](0))_0 = x$ depends only on a finite initial segment of φ. But every neighborhood of φ contains elements of F, and the assumption on σ implies (by a roundabout argument) that if $\overline{\psi}(x) = \overline{\varphi}(x)$ and $\psi \in \mathsf{F}$ then $(\{\sigma\}[\psi](0))_0 = x$ also. This provides the basis for informal bar induction.

For Kleene's ${}_{\mathsf{S}}$realizability, F is the class of all functions. To see that Markov's Principle is not ${}_{\mathsf{S}}$realizable, assume σ ${}_{\mathsf{S}}$realizes $\forall\alpha(\neg\forall\mathrm{x}\neg(\alpha(\mathrm{x}) = 0) \to \exists\mathrm{x}(\alpha(\mathrm{x}) = 0))$. Then $\tau = \{\sigma\}[\lambda t.1]$ ${}_{\mathsf{S}}$realizes $\neg\forall\mathrm{x}\neg(\alpha(\mathrm{x}) = 0) \to \exists\mathrm{x}(\alpha(\mathrm{x}) = 0)$ when α is interpreted by $\lambda t.1$, and there is a recursive θ which agrees with the hypothesis, so $(\{\tau\}[\theta](0))_0 = x$ is determined by $\overline{\lambda t.1}(z)$ for some z. If $\varphi(y) = 1$ for $y \leq x+z$ but $\varphi(x+z+1) = 0$, then θ ${}_{\mathsf{S}}$realizes the hypothesis $\neg\forall\mathrm{x}\neg(\alpha(\mathrm{x}) = 0)$ when α is interpreted by φ, but $\{\tau\}[\theta]$ does not ${}_{\mathsf{S}}$realize the conclusion.

The relativized version was developed in [37] (for F the class of all recursive functions) in order to prove $\forall\alpha\neg\neg\mathrm{GR}(\alpha)$ consistent with the intuitionistic theory and with Vesley's Schema [42], which entails Brouwer's creating subject counterexamples. For other applications, first observe that ordinary function-realizability provides a classical proof that the intuitionistic theory **FIM** is consistent with **PA**,

[46] [30] Corollary 9.9. The circularity in Brouwer's "proof" of his Bar Theorem is analyzed in Chapter 6.

since classically every *arithmetical* instance of the law of excluded middle is realized by some function.

If A(x) is any arithmetical predicate with only x free, let $\mathsf{F}[\mathrm{A(x)}]$ be the collection of all functions classically recursive in the intended interpretation of A(x). Then both $\forall\alpha\neg\neg\exists e\exists\beta[\forall x(\beta(x) = 0 \leftrightarrow A(x))\ \&\ \forall x\exists y[T_1(e, x, \overline{\beta}(y))\ \&\ U(\overline{\beta}(y)) = \alpha(x)]]$ and $\forall x(A(x) \vee \neg A(x))$ are ${}^{\mathsf{F}[A(x)]}$realizable and hence consistent with **FIM**. Thus it is possible to interpret the constructively undetermined part of the intuitionistic continuum as classically recursive in the constructively determined part, which could include the characteristic functions of arbitrarily complicated arithmetical relations. Of course, it is impossible to assign gödel numbers (or even relative gödel numbers) continuously, so the $\neg\neg\exists e$ cannot consistently be replaced by $\exists e$.

4.7. **A glimpse at today and tomorrow.** His philosophical ideas were Brouwer's motivation for the intuitionistic reconstruction of mathematics. However, in an irony of history, the interest in intuitionistic thought (except among those holding similar philosophical ideas) springs today mostly from logic and computer science.

The B-H-K interpretation found a precise implementation in Kleene's notion of realizability, which relates it to the theory of recursive functions; but also in connection with the proof systems of Gentzen, it led to the Curry-Howard isomorphism, which correlates λ-terms and finally programs with proofs. The intuitionistic theory of types of Per Martin-Löf belongs in the same framework. The polymorphic λ-calculus or system F of Girard and the logic of constructions of Coquand and Huet (on which was based the functional language proof checker Coq, used recently (2004) to verify the correctness of the solution to the 4 color problem), are applications of the preceding. A detailed description of the development and further influence of these ideas appears in the article "From constructivism to computer science" by A. S. Troelstra, [49].

Another source of interest is category theory, where it was discovered that important spaces such as topoi are models of intuitionistic logic. In the article [41] by J. van Oosten one can find a sketch of this line of development of the theory.

Various logical and mathematical systems and semantics were developed in connection with intuitionistic logic. As examples we mention Kripke semantics and constructive versions of Zermelo - Fraenkel set theory, like those of J. Myhill and P. Aczel; also the constructive development of a large part of analysis by E. Bishop and his school, and the recursive constructive mathematics of the Russian school associated with A. A. Markov.

In each case, intuitionistic mathematics constitutes a rich source of mathematical ideas. These can either be developed within classical mathematics to produce proper subtheories of particular classical theories, or be considered as extending classical mathematics,[47] offering different definitions and refinements of logical and mathematical concepts, with all that this implies.

References

[1] Benacerraf, P. and Putnam, H., *Philosophy of Mathematics. Selected Readings*, Prentice Hall 1964.

[2] Beth, E.W., *The foundations of mathematics*, North-Holland, Amsterdam 1959.

[47]As specific cases of the mathematical implementation of this viewpoint we mention [38, 39]

[3] Borel, E., *Sur les principes de la théorie des ensembles*, in G.Castelnuovo , ed., *Atti dei IV Congresso Internazionale dei Matematici, Roma, 6-11 Aprile 1908*, Vol.II, 15-17.

[4] Buss, S., ed., *Handbook of Proof Theory*, Elsevier, Amsterdam 1998.

[5] Brouwer, L. E. J., *On the foundations of mathematics*, English translation, in Heyting, ed., *L. E. J. Brouwer Collected Works, Vol 1*, 11-101.

[6] Brouwer, L. E. J., *The untrustworthiness of the logical principles*, English translation, in Heyting, ed., *L. E. J. Brouwer Collected Works, Vol 1*, 102-104.

[7] Brouwer, L. E. J., *Intuitionism and formalism*, translated by A. Dresden, Bull. Amer. Math. Soc. 20 (1913), 81-96. Reprinted in Benacerraf and Putnam, eds., *Philosophy of Mathematics*, 66-77.

[8] Brouwer, L. E. J., *Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten. Erster Teil: Allgemeine Mengenlehre*, in Heyting, ed., *L. E. J. Brouwer Collected Works, Vol 1*, 150-190.

[9] Brouwer, L. E. J., *On the domains of definition of functions*, English trans. in Van Heijenoort, ed., *From Frege to Gödel*, 446-463.

[10] Brouwer, L. E. J., *Die Struktur des Kontinuums*, in Heyting, ed., *L. E. J. Brouwer Collected Works, Vol 1*, 429-440.

[11] *Brouwer's Cambridge Lectures on Intuitionism*, edited by D. van Dalen, Cambridge University Press (1981)

[12] Dalen, D. van, *Mystic, Geometer, and Intuitionist. The Life of L. E. J. Brouwer*, Clarendon press, Oxford, 1999.

[13] Gentzen, G., *Untersuchungen über das logische Schliessen*, Math. Zeitschrift 39 (1934-5), 176-210, 405-431.

[14] Gentzen, G., *Über das Verhältnis zwischen intuitionistischer und klassischer Logik*, accepted by Math. Annalen (1933) but withdrawn; published in Arch. Math. Logik 16 (1974), 119-132. English trans. in Szabo, M., ed., *Gentzen: Collected Papers*, North-Holland, Amsterdam (1969), 53-67.

[15] Glivenko, V., *Sur la logique de M. Brouwer*, Akadémie Royale de Belgique, Bulletin de la Classe des Sciences (5) 14 (1928), 225-228.

[16] Glivenko, V., *Sur quelques points de la logique de M. Brouwer*, Acad. Roy. Belg., Bull. Sci. (5) 15 (1929), 183-88. English translation (*On some points of the logic of Mr. Brouwer*) by A. Rocha in Mancosu, P., ed., *From Brouwer to Hilbert*, 301-305.

[17] Gödel, K., *Eine interpretation des intuitionistischen Aussagenkalkúls*, Ergebnisse eines math. Kolloq. 4 (1933), 39-40.

[18] Heijenoort, J. van, ed., *From Frege to Gödel. A Source Book in Mathematical Logic 1879-1931*, Harvard University Press 1967.

[19] Heyting, A., *Die formalen Regeln der intuitionistischen Mathematik* I, Sitzungsberichte der Preuss. Akad. von Wissenschaften, physikalisch-mathematische Klasse (1930), 42-56. English translation (*The formal rules of intuitionistic logic*) in Mancosu, ed., *From Brouwer to Hilbert*, 311-327.

[20] Heyting, A., *Die formalen Regeln der intuitionistischen Mathematik* II, Sitz. Preuss. Akad. Wiss., phys.-math. Kl. (1930), 57-71.

[21] Heyting, A., *Die formalen Regeln der intuitionistischen Mathematik* III, Sitz. Preuss. Akad. Wiss., phys.-math. Kl. (1930), 158-169.

[22] Heyting, A., *Sur la logique intuitionniste*, Acad'emie Royale de Belgique, Bull. 16 (1930), 957-63. English trans. (*On intuitionistic logic*) by A. Rocha in Mancosu, ed., *From Brouwer to Hilbert*, 306-310.

[23] Heyting, A., Ed., *L. E. J. Brouwer Collected Works, Vol. 1: Philosophy and Foundations of Mathematics*, North-Holland/American Elsevier 1975.

[24] Hilbert, D. and Bernays, P., *Grundlagen der Mathematik*, Vol. 1 (1934), Vol. 2 (1939).

[25] Hesseling, D., *Gnomes in the Fog. The Reception of Brouwer's Intuitionism in the 1920s*, Birkhaüser Verlag 2003.

[26] Howard, W. and Kreisel, G., *Transfinite induction and bar induction of types zero and one, and the role of continuity in intuitionistic analysis*, Jour. Symbolic Logic 31 (1966) 325-358.

[27] Kino, A., Myhill, J. and Vesley, R., eds., *Intuitionism and Proof Theory*, North-Holland, Amsterdam, 1970.

[28] Kleene, S. C., *On the interpretation of intuitionistic number theory*, Jour. Symbolic Logic 10 (1945), 109-124.

[29] Kleene, S. C., *Introduction to Metamathematics*, North-Holland, Amsterdam, 1952.
[30] Kleene, S. C. and Vesley, R. E., *The Foundations of Intuitionistic Mathematics, Especially in Relation to Recursive Functions*, North-Holland, Amsterdam, 1965.
[31] Kleene, S. C., *Formalized recursive functionals and formalized realizability*, Memoirs, no. 89, American Mathematical Society, 1969.
[32] Kolmogorov, A., *On the principle of excluded middle*, translated from the Russian in van Heijenoort, ed. *From Frege to Gödel*, 414-437.
[33] Kolmogorov, A., *Zur Deutung der intuitionistischen Logik*, Math. Zeitschrift 35 (1932), 58-65. English trans. (*On the interpretation of intuitionistic logic*) in Mancosu, ed., *From Brouwer to Hilbert*, 328-334.
[34] Kreisel, G., *The non-derivability of* $\neg(x)A(x) \rightarrow (Ex)\neg A(x)$, *A primitive recursive, in intuitionistic formal systems*, Jour. Symb. Logic 23 (1958), 456-457.
[35] Loeb, I., *Equivalents of the (Weak) Fan Theorem*, Annals of Pure and Applied Logic 132 (2005), 51-66.
[36] Mancosu, P., ed., *From Brouwer to Hilbert. The Debate on the Foundations of Mathematics in the 1920s*, Oxford University Press, 1998.
[37] Moschovakis, J. R., *Can there be no non-recursive functions*, Jour. Symb. Logic 36 (1971), 309-315.
[38] Moschovakis, J. R., *A classical view of the intuitionistic continuum*, Annals of Pure and Applied Logic 81 (1996) 9-24.
[39] Moschovakis, J. R., *Classical and constructive hierarchies in extended intuitionistic analysis*, Jour. Symb. Logic 64 (2003), 1015-1043.
[40] Nelson, D., *Recursive functions and intuitionistic number theory*, Trans. Amer. Math. Soc. 61 (1947), 307-368.
[41] Oosten, J. van, *Realizability: a historical essay*, Math. Struct. Comp. Sci 12 (2002), 239-263.
[42] Vesley, R. E., *A palatable substitute for Kripke's schema*, in Kino, Myhill, Vesley, ed., *Intuitionism and Proof Theory*, 197-207.
[43] Stigt, W. P. van, *Brouwer's Intuitionism*, North-Holland, Amsterdam (1990).
[44] Peano, G., *The principles of arithmetic, presented by a new method*, English trans. in Van Heijenoort, ed., *From Frege to Gödel*, 83-97.
[45] Troelstra, A. S., *The scientific work of A. Heyting*, Compositio Mathematica 20 (1968), 3 - 12.
[46] Troelstra, A. S., *Note on the fan theorem*, Jour. Symb. Logic 39 (1974), 584-596.
[47] Troelstra, A. S., *History of constructivism in the 20th century*, Univ. Amsterdam preprint ML-91-05, 1991.
[48] Troelstra, A. S., *Realizability*, in S. Buss, ed., *Handbook of Proof Theory*, 407-473.
[49] Troelstra, A. S., *From Constructivism to Computer Science*, Theoretical Computer Science 211, 233–252. Text of the lecture held on occasion of receiving the F.L. Bauer-prize. Preprint: ILLC Research report CT-96-02.
[50] Troelstra, A. S. and Dalen, D. van, *Constructivism in Mathematics* I and II, North-Holland, Amsterdam, 1988.