ITERATED DEFINABILITY, LAWLESS SEQUENCES AND BROUWER'S CONTINUUM

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Abstract. The research on which this article is based was motivated by the wish to find a model of Kreisel's lawless sequence axioms in which the lawlike and lawless sequences form disjoint, inhabited, well-defined classes within Brouwer's continuum. The original results, reported as they developed in four papers over a period of ten years from 1986 to 1996, have so far lacked a reader-friendly presentation. Since the question of absolute definability is related to the subject of these Bristol Workshops, I offer here a straightforward exposition of the final model and formal system with axioms for numbers, lawlike sequences, and arbitrary choice sequences. A choice sequence is defined to be lawless if it satisfies an extensional (un)predictability condition from which extensional versions of Kreisel's axioms of open data and strong continuous choice follow. The law of excluded middle can be assumed for properties of lawlike and independent lawless sequences only, while Brouwer's continuity principle applies to properties of all choice sequences.

Iterating definability, quantifying over numbers and over lawlike and independent lawless sequences, yields a classical model of the lawlike sequences with a natural wellordering. Under the (classically consistent and intuitionistically plausible) assumption that the closure ordinal of the iteration is countable, a realizability interpretation establishes the consistency of a common extension **FIRM**($\prec$) of classical analysis **R** and Kleene's intuitionistic analysis **FIM**. Lawlike sequences behave classically, while the lawless sequences form a disjoint, Baire comeager collection of choice sequences, of classical measure zero. Thus Brouwer's continuum can be understood as a relatively chaotic expansion of a completely determined, well-ordered classical continuum.

1. Introduction

Gödel argued in [2] that the unprovability, in any consistent recursively enumerable extension of the arithmetic of addition and multiplication, of the consistency of that system strongly suggests that "the human mind ... infinitely surpasses the powers of any finite machine." Implicit in this argument is the assumption that arithmetic is in fact consistent, and that the human mind can recognize this fact.

Brouwer, for whom all mathematics was open-ended, probably would have said that the consistency of intuitionistic arithmetic is a trivial consequence of its truth. Gödel's negative translation showed that "*the system of intuitionistic arithmetic and number theory is only apparently narrower than the classical one, and in truth contains it, albeit with a somewhat deviant interpretation.*"[1]

This is an expanded version of a talk given on March 31, 2013 at the Second Bristol Workshop on the Limitations and Scope of Mathematical Knowledge. I thank Leon Horsten and Philip Welch for organizing the conference, the Templeton Foundation for sponsoring it, and James Appleby for asking the questions which inspired this exposition.

[1]From the translation of [1] on p. 295 of Volume I of Gödel's collected works.

Formally, intuitionistic arithmetic is a proper subsystem of classical arithmetic. But intuitionistic analysis **FIM** includes a continuity principle incompatible with classical predicate logic. Kleene's function-realizability interpretation, a recursive implementation of the Brouwer-Heyting-Kolmogorov explanation of the intuitionistic connectives and quantifiers, proves **FIM** consistent relative to its classically correct subsystem **B**; so intuitionistic and classical analysis are equiconsistent.

The negative interpretation does not extend to analysis because the negative translation of the axiom of countable choice is not an intuitionistic principle. It is consistent with **FIM** by realizability (using a classical argument), and so classical analysis **R** has a "somewhat deviant" negative interpretation in a consistent extension of **FIM**.

Since classical arithmetic is consistent with **FIM**, one might hope for a different consistent extension of intuitionistic analysis containing classical analysis (not just its negative interpretation) as a part. In [4] Kleene defended his choice to formalize **FIM** using only variables over numbers and over arbitrary choice sequences, since the general recursive functions were the only constructive functions he needed and those could be coded by numbers. For a quasi-intuitionistic system including classical analysis a third sort of variable is indicated, as even the class of all hyperarithmetical sequences fails to satisfy the (classically and intutionistically correct) bar theorem, and if every choice sequence was a classical object the continuity principle would fail.

Brouwer's notion of "lawlike sequence" (choice sequence all of whose values are determined in advance) provides a solution. The lawlike sequences can play the role of the classical objects within the intuitionistic continuum, and this role can be formalized in a three-sorted axiomatic system **FIRM** containing **R** and **FIM** as two-sorted subsystems, with extensional versions of Kreisel's "lawless sequence" axioms for a class of choice sequences defined in terms of the lawlike sequences. Consistency is established for a stronger system **FIRM($\prec$)** assuming an ordinal-definable, definably well-ordered collection of lawlike sequences is countable. For an intuitionist, the assumption that there are at most countably many lawlike sequences needs no justification.

2. Choice sequences

2.1. **Brouwer's continuum.** For the intuitionistic theory of the continuum, Brouwer's concept of "choice sequence" replaces the classical notion of an arbitrary completed infinite sequence of natural numbers. Choice sequences are generated by more or less freely choosing one integer after another. At each stage, the chooser may or may not specify restrictions (consistent with those already made) on future choices.

Since the first n values of a choice sequence α may be the only information available at the nth stage of its construction, every function defined on *all* choice sequences must be continuous, so for example if $A(\alpha)$ is $\forall x(\alpha(x) = 0)$ then $\neg\forall\alpha(A(\alpha) \vee \neg A(\alpha))$ holds.

Any choice sequence *all* of whose values are determined in advance according to some fixed law is called "lawlike" or a "sharp arrow." Brouwer did not specify which laws may determine lawlike sequences, but evidently every lawlike sequence is definable in some sense.

2.2. **The problem of defining "definability".** Considering whether or not it is possible to define the concept of absolute mathematical definability, Gödel [3] observed that constructibility is not a "satisfactory formulation of definability" because "quantification is admitted only with respect to constructible sets and not with respect to sets in general." Instead he proposed "definability by expressions containing names of ordinal numbers and logical constants, including quantification referring to sets," adding

> ... "definability in terms of ordinals" even if it is not an adequate formulation for "comprehensibility by our mind", is at least an adequate formulation in an absolute sense for a closely related property ..., namely the property of "being formed according to a law" as opposed to "being formed by a random choice of the elements".

This idea can be adapted to define "lawlike sequence" by iterating definability, with quantification over numbers and lawlike sequences and (restricted) quantification over arbitrary choice sequences, to build a classical model of a three-sorted theory. The integers of the model are the standard integers, the arbitrary choice sequences of the model are all the classical one-place number-theoretic functions, and the lawlike sequences of the model form a definably well-ordered subclass of the choice sequences.

2.3. **"Lawlike" vs. "lawless" sequences.** Kreisel [7] proposed axioms for numbers, "constructive" functions, and intensionally "lawless" sequences in which "the *simplest kind of restriction on restrictions is made, namely some finite initial segment of values is prescribed, and, beyond this, no restriction is to be made"*. His constructive functions satisfy countable and dependent choice. Equality of lawless sequences is decidable. Distinct lawless sequences are independent and satisfy the axiom of open data and a strong form of continuous choice ("the only problematic axiom"). There are no variables over arbitrary choice sequences.

Troelstra [13] pointed out that Kreisel's problematic axiom entailed an "extension principle": every continuous number-valued functional defined on the lawless sequences has a continuous extension to all sequences. To justify his extension principle philosophically he introduced a conceptual abstraction operator allowing *any* choice sequence to be viewed as lawless. He also filled in the details of Kreisel's proof that every sentence of his two-sorted language is formally equivalent to one without lawless sequence variables, so "lawless sequences can be regarded as a figure of speech." Troelstra used the term "lawlike" instead of "constructive," and noted that the lawlike sequence variables *may* be interpreted as ranging over "the classical universe of sequences" ([13] p. 4).

Kreisel's notion of lawless sequence collapsed because he identified independence with inequality. The collapse can be avoided by taking α, β to be independent lawless sequences if and only if their fair merge $[\alpha, \beta]$ is lawless. Then Kreisel's axioms can be forced to hold by defining "lawless" relative to a given wellordered collection of lawlike sequences whose intended interpretation is countable.

3. The formal systems RLS($\prec$) and FIRM($\prec$)

RLS($\prec$) is the system for which a classical model will be obtained by iterating definability, and **FIRM**($\prec$) is the quasi-intuitionistic extension of **RLS**($\prec$) whose consistency follows by realizability.

3.1. **The three-sorted language $\mathcal{L}(\prec)$.** The language, extending the two-sorted language of [6] and [5], contains three sorts of variables with or without subscripts, also used as metavariables:

$i, j, k, \ldots, p, q, w, x, y, z$ over natural numbers,
a, b, c, d, e, g, h over lawlike sequences,
$\alpha, \beta, \gamma, \ldots$ over arbitrary choice sequences;

finitely many constants $\mathrm{f}_0 = 0$, $\mathrm{f}_1 = {}'$ (successor), $\mathrm{f}_2 = +$, $\mathrm{f}_3 = \cdot$, $\mathrm{f}_4 = \exp$, $\mathrm{f}_5, \ldots, \mathrm{f}_p$ for primitive recursive functions and functionals; two binary predicate constants $=, \prec$; Church's λ denoting function abstraction; parentheses (,) denoting function application; and the logical symbols $\&, \vee, \rightarrow, \neg$ and quantifiers $\forall, \exists$ over each sort of variable.

Terms (of type 0) and *functors* (of type 1) are defined inductively. Number variables and 0 are terms. Sequence variables of both sorts, and unary function constants, are functors. If f_i is a k_i, m_i-ary function constant, $u_1, \ldots, u_{k_i}$ are functors and $t_1, \ldots, t_{m_i}$ are terms, then $\mathrm{f}_i(u_1, \ldots, u_{k_i}, t_1, \ldots, t_{m_i})$ is a term. If u is a functor and t is a term then $(u)(t)$ is a term. If t is a term then $\lambda x(t)$ (also written $\lambda x.t$) is a functor.

Prime formulas are of the form $s = t$ where s, t are terms, or $u \prec v$ where u, v are functors. If u, v are functors then $u = v$ abbreviates $\forall x u(x) = v(x)$. Composite formulas are formed as usual, with parentheses determining scopes.

Terms and functors with no occurrences of arbitrary choice sequence variables are *R-terms* and *R-functors* respectively. Formulas with no free occurrences of arbitrary choice sequence variables are *R-formulas*.

3.2. **Axioms and rules for 3-sorted intuitionistic predicate logic.** The logical basis is intuitionistic three-sorted predicate logic, as in [6] but with additional rules and axiom schemas for the lawlike sequence variables:

9^R. $C \rightarrow A(b)$ / $C \rightarrow \forall b A(b)$ if b is not free in C.
10^R. $\forall b A(b) \rightarrow A(u)$ if u is an R-functor free for b in $A(b)$.
11^R. $A(u) \rightarrow \exists b A(b)$ if u is an R-functor free for b in $A(b)$.
12^R. $A(b) \rightarrow C$ / $\exists b A(b) \rightarrow C$ if b is not free in C.

$\forall a \exists! \beta (\forall x\ a(x) = \beta(x))$ is a theorem, so every lawlike sequence is a choice sequence.[2]

3.3. **Axioms for 3-sorted intuitionistic number theory:** Equality axioms assert that $=$ is an equivalence relation and $x = y \rightarrow \alpha(x) = \alpha(y)$ (so $x = y \rightarrow a(x) = a(y)$). For terms $r(x), t$ with t free for x in $r(x)$, the λ-reduction schema is

$$(\lambda x.r(x))(t) = r(t),$$

where $r(t)$ is the result of substituting t for all free occurrences of x in $r(x)$.

The mathematical axioms include the assertions that 0 ($= \mathrm{f}_0$) is not a successor and the successor function ($= \mathrm{f}_1$) is one-to-one, the defining equations for the primitive recursive function and functional constants $\mathrm{f}_2, \ldots, \mathrm{f}_p$ ([6], [5]) and the mathematical induction schema extended to $\mathcal{L}(\prec)$. For the countable axiom of choice

$$\mathrm{AC}_{01}. \quad \forall x \exists \alpha A(x, \alpha) \rightarrow \exists \alpha \forall x A(x, \lambda y.\alpha(2^x \cdot 3^y))$$

the x must be free for α in $A(x, \alpha)$.

[2]The ! expresses uniqueness: $\exists! \beta A(\beta)$ abbreviates $\exists \beta A(\beta)\ \&\ \forall \beta \forall \gamma (A(\beta)\ \&\ A(\gamma) \rightarrow \beta = \gamma)$, and $\exists! x A(x)$ abbreviates $\exists x A(x)\ \&\ \forall x \forall y (A(x)\ \&\ A(y) \rightarrow x = y)$.

Finite sequences are coded primitive recursively as in [6]. Let $\langle x_0, \ldots, x_k \rangle = \Pi_0^k p_i^k$ where p_i is the ith prime with $p_0 = 2$, and $(y)_i$ be the exponent of p_i in the prime factorization of y. Let $Seq(y)$ abbreviate $\forall i < lh(y)\ (y)_i > 0$ where $lh(y)$ is the number of nonzero exponents in the prime factorization of y. If $k \geq 0$ then $\langle x_0 + 1, \ldots, x_k + 1 \rangle$ codes the finite sequence $(x_0, \ldots, x_k)$, and $\langle\,\rangle = 1$ codes the empty sequence. If $Seq(y)$ and $Seq(z)$ then $y * z$ codes the concatenation of the sequences coded by y and z.

The finite initial segment of length n of a choice sequence α is coded by $\overline{\alpha}(n)$, where $\overline{\alpha}(0) = 1$ and $\overline{\alpha}(n+1) = \langle \alpha(0)+1, \ldots, \alpha(n)+1 \rangle$. Other useful abbreviations are $\alpha \in w$ for $\overline{\alpha}(lh(w)) = w$, and $w \in \overline{\alpha}$ for $\exists n(w = \overline{\alpha}(n))$. If $Seq(w)$ then $w * \alpha = \beta$ where $\beta \in w$ and $\beta(lh(w) + n) = \alpha(n)$.

3.4. **Lawless sequences, restricted quantification, and lawlike comprehension.** Intuitively, a lawless sequence should not be predictable by any lawlike process, but this negative condition is not enough to satisfy Kreisel's axioms. Instead, call a sequence β a *predictor* if $\forall w(Seq(w) \rightarrow Seq(\beta(w)))$, and call a choice sequence α *lawless* if every lawlike predictor correctly predicts α somewhere. Since each prediction affects only finitely many values, this positive condition leaves room for (indeed, guarantees) plenty of chaotic behavior *if there are only countably many lawlike predictors.* From the intuitionistic point of view this assumption needs no justification; but there is no lawlike enumeration of the lawlike sequences.

Formally, let $RLS(\alpha)$ abbreviate

$$\forall b[\forall w(Seq(w) \rightarrow Seq(b(w))) \rightarrow \exists x\ \alpha \in \overline{\alpha}(x) * b(\overline{\alpha}(x))].$$

Note that Troelstra's extension principle fails, since $\forall \alpha(RLS(\alpha) \rightarrow \exists n \alpha(n) = 1)$ but the function assigning to each lawless α the least n such that $\alpha(n) = 1$ cannot be extended continuously to all choice sequences.

Two lawless sequences α, β are *independent* if their fair merge $[\alpha, \beta]$ is lawless, and similarly for $\alpha_0, \ldots, \alpha_k$, where $[\alpha_0, \ldots, \alpha_k]((k+1)n + i) = \alpha_i(n)$ for $0 \leq i \leq k$ and all n. A *restricted* formula E has arbitrary choice sequence quantifiers only in subformulas of the form $\forall \alpha_0(RLS([\alpha_0, \ldots, \alpha_k]) \rightarrow A)$ and $\exists \alpha_0(RLS([\alpha_0, \ldots, \alpha_k])\ \&\ A)$ where no arbitrary choice sequence variables but $\alpha_0, \ldots, \alpha_k$ occur free in A.

There is a lawlike function-comprehension schema

$$\mathrm{AC}_{00}^{R}!. \quad \forall x \exists ! y A(x, y) \rightarrow \exists b \forall x A(x, b(x))$$

where $A(x, y)$ is any restricted R-formula and b is free for y in $A(x, y)$. By this axiom, the lawlike sequences are closed under "recursive in."

3.5. **Axioms for lawless sequences.** These are Kreisel's and Troelstra's axioms from [7] and [13], adapted to Kleene's convention for coding continuous functions, with inequality of lawless sequences replaced by independence. There are two *density axioms*:

$$\mathrm{RLS1}. \quad \forall w[Seq(w) \rightarrow \exists \alpha(RLS(\alpha)\ \&\ \alpha \in w)].$$

$$\mathrm{RLS2}. \quad \forall \alpha[RLS(\alpha) \rightarrow \forall w[Seq(w) \rightarrow \exists \beta(RLS([\alpha, \beta])\ \&\ \beta \in w)]].$$

Kreisel's principle of *open data* is stated as follows, on condition that $A(\alpha)$ is restricted and has no other arbitrary choice sequence variables free, and β is free for α in $A(\alpha)$:

$$\mathrm{RLS3}. \quad \forall \alpha[RLS(\alpha) \rightarrow (A(\alpha) \rightarrow \exists y \forall \beta(RLS(\beta)\ \&\ \beta \in \overline{\alpha}(y) \rightarrow A(\beta)))].$$

Effective continuous choice for lawless sequences is the schema

$$\text{RLS4.}\quad \forall\alpha[RLS(\alpha) \rightarrow \exists b A(\alpha, b)] \rightarrow \exists e \exists b \forall\alpha[RLS(\alpha) \rightarrow \exists! y e(\overline{\alpha}(y)) > 0 \;\&\; \forall y(e(\overline{\alpha}(y)) > 0 \rightarrow A(\alpha, \lambda x.\, b(\langle e(\overline{\alpha}(y)) \dot{-} 1, x\rangle))))]$$

where $A(\alpha, b)$ is restricted with no arbitrary choice sequence variables but α free, and e, y, α are free for b in $A(\alpha, b)$.

3.6. **Wellordering the lawlike sequences.** Axioms W0-4 assert that the lawlike sequences are well-ordered by $\prec$, and W5 weakly specifies the domain of $\prec$:

$$\text{W0.}\quad \alpha = \beta \;\&\; \alpha \prec \gamma \rightarrow \beta \prec \gamma \quad \text{and} \quad \beta = \gamma \;\&\; \alpha \prec \beta \rightarrow \alpha \prec \gamma.$$

$$\text{W1.}\quad \forall a \forall b[a \prec b \rightarrow \neg b \prec a].$$

$$\text{W2.}\quad \forall a \forall b \forall c[a \prec b \;\&\; b \prec c \rightarrow a \prec c].$$

$$\text{W3.}\quad \forall a \forall b[a \prec b \;\vee\; a = b \;\vee\; b \prec a].$$

$$\text{W4.}\quad \forall a[\forall b(b \prec a \rightarrow A(b)) \rightarrow A(a)] \rightarrow \forall a A(a),$$

where $A(a)$ is any restricted R-formula in which b is free for a.

$$\text{W5.}\quad \alpha \prec \beta \rightarrow \neg\forall a \forall b \neg(\alpha = a \;\&\; \beta = b).$$

The double negation in W5 is essential. Replacing $\neg\forall a \forall b \neg$ by $\exists a \exists b$ would destroy the realizability model.

3.7. **Restricted LEM, the axiom of closed data and lawlike countable choice.** The final axiom schema for **RLS(≺)** is the *restricted law of the excluded middle*:

$$\text{RLEM.}\quad \forall\alpha(RLS(\alpha) \rightarrow A(\alpha) \vee \neg A(\alpha))$$

for $A(\alpha)$ restricted and with no other arbitrary choice sequence variables free. Lawless sequences exist by RLS1, so $A \vee \neg A$ follows for A without arbitrary choice sequence variables. Thus the logic of the two-sorted subsystem with only number and lawlike sequence variables, omitting W5 and with the α, β, γ in W0 and the axioms for 3-sorted intuitionistic number theory replaced by a, b, c respectively, is classical.

By an easy argument, RLS3 and the restricted LEM entail the following principle of *closed data* with the same restrictions on $A(\alpha)$ as for RLS3:

$$\text{RLS5.}\quad \forall\alpha[RLS(\alpha) \rightarrow (\forall y \exists\beta(RLS(\beta) \;\&\; \beta \in \overline{\alpha}(y) \;\&\; A(\beta)) \rightarrow A(\alpha))].$$

In an intuitionistic subsystem obtained by omitting RLEM, RLS5 may be taken as an additional axiom schema.

For restricted R-formulas $A(x, a)$ the lawlike comprehension schema entails

$$\text{AC}^R_{01}!.\quad \forall x \exists! a A(x, a) \rightarrow \exists b \forall x A(x, \lambda y.b(2^x \cdot 3^y)),$$

with the obvious conditions on the variables. By the wellordering axioms, with the same conditions this schema can be strengthened to lawlike countable choice

$$\text{AC}^R_{01}.\quad \forall x \exists a A(x, a) \rightarrow \exists b \forall x A(x, \lambda y.b(2^x \cdot 3^y)).$$

Lawlike classical analysis is the subsystem **R** of **RLS(≺)** obtained by restricting the language to number and lawlike sequence variables as above, omitting $\prec$ and its axioms, and replacing RLEM by $A \vee \neg A$ and $\text{AC}^R_{00}!$ by AC^R_{01}.

3.8. **Brouwer's bar theorem and Troelstra's generalized continuous choice.** Two axiom schemas, in addition to those already stated (in the two-sorted language $\mathcal{L}_{\mathbf{FIM}}$ without $\prec$ and with number and arbitrary choice sequence variables only), characterize Kleene and Vesley's full system **FIM** of intuitionistic two-sorted number theory in [6] and [5]. These specifically intuitionistic axiom schemas are Brouwer's "bar theorem" and strong continuous choice, stated now for $\mathcal{L}(\prec)$.

Kleene gave four versions of the bar theorem which are interderivable using $\mathrm{AC}_{00}!$ (a consequence of AC_{01}). The version adopted here assumes a thin bar:

$$\text{BI!.}\quad \forall\alpha\exists!xR(\overline{\alpha}(x))\ \&\ \forall w(Seq(w)\ \&\ R(w)\rightarrow A(w))\ \&\\ \forall w(Seq(w)\ \&\ \forall nA(w*\langle n+1\rangle)\rightarrow A(w))\rightarrow A(\langle\,\rangle).$$

Kleene's strong version of Brouwer's principle of continuous choice is

$$\mathrm{CC}_{11}.\quad \forall\alpha\exists\beta A(\alpha,\beta)\rightarrow\exists\sigma\forall\alpha(\forall n\exists!x\sigma(\langle n+1\rangle*\overline{\alpha}(x))\neq 0\ \&\\ \forall\beta(\forall n\exists x\sigma(\langle n+1\rangle*\overline{\alpha}(x))=\beta(n)+1\rightarrow A(\alpha,\beta))),$$

which can be abbreviated by $\forall\alpha\exists\beta A(\alpha,\beta)\rightarrow\exists\sigma\forall\alpha(\{\sigma\}[\alpha]\downarrow\ \&\ A(\alpha,\{\sigma\}[\alpha]))$. The necessary restrictions on the bound variables are obvious and will not be stated.

Troelstra [12] found a strengthening of Brouwer's principle which precisely characterizes the function-realizability notion of [6]. Troelstra's *generalized continuous choice* schema can be abbreviated by

$$\mathrm{GC}_{11}.\quad \forall\alpha(A(\alpha)\rightarrow\exists\beta B(\alpha,\beta))\rightarrow\exists\sigma\forall\alpha(A(\alpha)\rightarrow\{\sigma\}[\alpha]\downarrow\ \&\ B(\alpha,\{\sigma\}[\alpha])),$$

where $A(\alpha)$ must be *almost negative* (i.e. contain no $\vee$, and no $\exists$ except immediately before an equation between terms). The maximal system considered in this article, with all schemas extended to the language $\mathcal{L}(\prec)$, is

$$\mathbf{FIRM}(\prec)=\mathbf{RLS}(\prec)+\text{BI!}+\mathrm{GC}_{11}.$$

3.9. **Classical and intuitionistic analysis as subsystems of FIRM($\prec$).** Four subsystems of the maximal system are of independent interest. The main theorem, that **FIRM($\prec$)** is consistent, depends on the assumption that its subsystem **RLS**($\prec$) has a classical model with just the standard integers and all the classical arbitrary choice sequences, but only countably many lawlike sequences.

The subsystem **FIRM** obtained by restricting the language to $\mathcal{L}$ (without $\prec$), dropping the axioms W0-W5, and replacing $\mathrm{AC}^R_{00}!$ by AC^R_{01} has lawlike classical analysis **R** (described in §3.7) and Kleene's intuitionistic analysis **FIM** and basic analysis **B** as two-sorted subsystems in the languages $\mathcal{L}_{\mathbf{R}}$ and $\mathcal{L}_{\mathbf{FIM}}$ ($=\mathcal{L}_{\mathbf{B}}$) respectively, where $\mathcal{L}_{\mathbf{R}}$ is obtained by excluding the arbitrary choice sequence variables from $\mathcal{L}$. For **FIM** Troelstra's GC_{11} is replaced by its consequence CC_{11}; **B** is the neutral subsystem of **FIM** with BI! but without continuous choice.

A classical version of the bar theorem for $\mathcal{L}_{\mathbf{R}}$, obtained from BI! by changing α to a and dropping the !, is provable in **R**.[3] Except for the choice of Latin rather than Greek letters for sequence variables, **R** is Kleene's **B** with classical logic.

[3] See p. 53 of [6]. The ! is essential for the intuitionistic version.

3.10. **Consistency of FIRM($\prec$).** This depends on two theorems and an assumption whose consistency follows from work of Levy [8]. Proofs of these theorems are outlined in the next two sections.

Theorem 1. Assuming a certain definably wellordered subset $\mathcal{R}$ of ω^ω (determined by iterating definability by restricted R-formulas) is countable, **RLS($\prec$)** has a classical model $\mathcal{M}(\prec_{\mathcal{R}})$ with $\mathcal{R}$ as the class of lawlike sequences. The class $\mathcal{RLS}$ of lawless sequences of the model is Baire comeager in ω^ω, and has classical measure 0.

Theorem 2. Under the same countability assumption, **FIRM($\prec$)** is consistent by a classical realizability interpretation. So the classical sequences can consistently be viewed as the lawlike elements of Brouwer's continuum, while the lawless sequences form a dense collection of choice sequences disjoint from the lawlike part.

4. Construction of the classical model and proof of Theorem 1

4.1. **Definability over $(A, \prec_A)$ by a restricted formula of $\mathcal{L}(\prec)$.** If $F(a_0, \ldots, a_{k-1})$ is a restricted formula of $\mathcal{L}(\prec)$ of the form $\forall x \exists ! y E(x, y, a_0, \ldots, a_{k-1})$ where x, y are all the distinct number variables free in E and $a_0, \ldots, a_{k-1}$ are all the distinct variables free in F in order of first free occurrence, and if $A \subseteq \omega^\omega$, $\prec_A$ is a binary relation on A, $\varphi \in \omega^\omega$ and $\psi_0, \ldots, \psi_{k-1} \in A$, then *$E$ defines φ over $(A, \prec_A)$ from $\psi_0, \ldots, \psi_{k-1}$* if and only if, when number variables range over ω, lawlike sequence variables range over A, choice sequence variables range over ω^ω, $\prec$ is interpreted by $\prec_A$, and $a_0, \ldots, a_{k-1}$ by $\psi_0, \ldots, \psi_{k-1}$:

(i) F is true, and
(ii) for all $x, y \in \omega$: $\varphi(x) = y$ if and only if $E(\mathbf{x}, \mathbf{y})$ is true.

Let **Def($A, \prec_A$)** be the class of all $\varphi \in \omega^\omega$ such that some restricted R-formula E of $\mathcal{L}(\prec)$ defines φ over $(A, \prec_A)$ from some $\psi_0, \ldots, \psi_{k-1}$ in A. Observe that $A \subseteq \mathrm{Def}(A, \prec_A)$, since $a(x) = y$ defines every $\varphi \in A$ over A from itself.

For the intended application, $\prec_A$ will be a wellordering of A determined inductively with the help of a fixed enumeration $E_0(x, y), E_1(x, y), \ldots$ of all restricted R-formulas of $\mathcal{L}(\prec)$ containing free no number variables but x, y, where $E_0(x, y) \equiv a(x) = y$. In the case that A is nonempty, $\mathrm{Def}(A, \prec_A)$ will be well-ordered by an end extension $\prec_A^*$ of $\prec_A$, defined as follows. For $\varphi, \theta \in \mathrm{Def}(A, \prec_A)$, set $\varphi \prec_A^* \theta$ if and only if $\Delta_A(\varphi) < \Delta_A(\theta)$ where $\Delta_A(\varphi)$ is the smallest tuple $(i, \psi_0, \ldots, \psi_{k-1})$ in the lexicographic ordering $<$ of $\bigcup_{k>o}(\omega \times A^k)$ determined by $<$ on ω and $\prec_A$ on A such that E_i defines φ over $(A, \prec_A)$ from $\psi_0, \ldots, \psi_{k-1}$. Then $\prec_A^*$ wellorders $\mathrm{Def}(A, \prec_A)$; and if $\varphi \in A$ then $\Delta_A(\varphi) = (0, \varphi)$, so $\prec_A$ is an initial segment of $\prec_A^*$. The case $A = \phi$ will be treated separately.

4.2. **The classical model $\mathcal{M}(\prec_{\mathcal{R}})$.** The next step is to define a structure $\mathcal{M}(\prec_{\mathcal{R}})$ for $\mathcal{L}(\prec)$ by iterating the process described in the previous section. More accurately, $\mathcal{M}(\prec_{\mathcal{R}}) = (\omega, \mathcal{R}, \omega^\omega, f_0, \ldots, f_p, =, \prec_{\mathcal{R}})$ where $f_0, \ldots, f_p$ are the standard interpretations of the primitive recursive functional constants $\mathrm{f}_0, \ldots, \mathrm{f}_p$ respectively, $=$ is equality of natural numbers, and $\mathcal{R}$ and $\prec_{\mathcal{R}}$ are defined as follows. The construction will guarantee that $\prec_{\mathcal{R}}$ is a wellordering of $\mathcal{R}$.

As above, let $E_0(x, y), E_1(x, y), \ldots$ be a fixed enumeration of all restricted R-formulas in the language $\mathcal{L}(\prec)$ containing free no number variables but x and y, where $E_0(x, y) \equiv a(x) = y$. Begin by defining $\mathcal{R}_0 = \phi$, $\prec_0 = \phi$, and $\mathcal{R}_1 = \mathrm{Def}(\mathcal{R}_0 \prec_0)$.

Since $RLS(\alpha)$ always holds and $u \prec v$ always fails when the lawlike sequence variables range over the empty set and $\prec$ is the empty relation, every restricted R-formula of $\mathcal{L}(\prec)$ with only the distinct number variables x, y free is equivalent, by an easy translation, over the structure $(\omega, \phi, \omega^\omega, f_0, \dots, f_p, =, \phi)$ to a formula of the two-sorted language $\mathcal{L}_{\mathrm{FIM}}$ with only x and y free. So $\mathcal{R}_1$ consists of all analytic sequences.

If φ is analytic, let $\Delta_0(\varphi)$ be the least $i \in \omega$ such that $E_i(x, y)$ is a formula of $\mathcal{L}_{\mathrm{FIM}}$ with only x, y free, and $E_i(x, y)$ defines φ. Then define $\varphi \prec_1 \psi$ if and only if $\varphi, \psi \in \mathcal{R}_1$ and $\Delta_0(\varphi) < \Delta_0(\psi)$. Evidently $\prec_1$ wellorders $\mathcal{R}_1$.

For $\zeta > 0$ define $\mathcal{R}_{\zeta+1} = \mathrm{Def}(\mathcal{R}_\zeta, \prec_\zeta)$ and $\prec_{\zeta+1} = \prec_\zeta^*$. At limit ordinals take unions. By cardinality considerations there is a least ordinal η_0 (with cardinality $\leq 2^{\aleph_0}$) such that $\mathcal{R}_{\eta_0} = \mathcal{R}_{\eta_0+1}$. Let $\mathcal{R} = \mathcal{R}_{\eta_0}$ and $\prec_{\mathcal{R}} = \prec_{\eta_0}$.

By transfinite induction, $\prec_\zeta$ wellorders $\mathcal{R}_\zeta$ and is an initial segment of $\prec_{\zeta+1}$ for each $\zeta < \eta_0$, so $\prec_{\mathcal{R}}$ wellorders $\mathcal{R}$. Hence $\mathcal{M}(\prec_{\mathcal{R}})$, the natural classical model in which lawlike sequence variables range over the subset $\mathcal{R}$ of ω^ω, satisfies axioms W0-W5.[4]

Since $\mathcal{M}(\prec_{\mathcal{R}})$ is a classical model, it evidently satisfies RLEM. Since $\mathcal{M}(\prec_{\mathcal{R}})$ satisfies axiom schema $\mathrm{AC}_{00}^R!$ by construction, the following lemmas hold.[5]

Lemma 1. If $\mathcal{M}(\prec_{\mathcal{R}})$ satisfies the axiom schema RLS3 of open data for a restricted formula $A(\alpha)$ with only α and a list Ψ of lawlike sequence variables free, then $\mathcal{M}(\prec_{\mathcal{R}})$ satisfies RLS3 for $\neg A(\alpha)$.

Proof. Recall that 0 is not a sequence code and 1 is the code of the empty sequence. Given $A(\alpha)$ with only α and the lawlike variables Ψ free, define restricted R-formulas

$$J(x, y) \equiv (Seq(x) \to Seq(y)) \ \&\ (y = 1 \to \forall\beta(RLS(\beta) \to (\beta \in x \to \neg A(\beta)))) \ \&$$
$$(y > 1 \to \forall\beta(RLS(\beta) \to (\beta \in x * y \to A(\beta)))),$$

$$K(x, y) \equiv J(x, y) \ \&\ \forall z(z < y \to \neg J(x, z)).$$

Assume $\mathcal{M}(\prec_{\mathcal{R}})$ satisfies RLS3 for $A(\alpha)$. Then under any assignment $\mathit{\Psi}$ of elements of $\mathcal{R}$ to Ψ: $\mathcal{M}(\prec_{\mathcal{R}}) \models_{\mathit{\Psi}} \forall x \exists! y K(x, y)$ and so $K(x, y)$ defines a lawlike predictor $\pi \in \mathcal{R}$ with the property that $\mathcal{M}(\prec_{\mathcal{R}}) \models_{\mathit{\Psi},\pi} \forall x K(x, b(x))$. If $\alpha \in \mathcal{RLS}$ there is a least n at which π correctly predicts α, so if $x = \overline{\alpha}(n)$ then $\alpha \in x * \pi(x)$ and $\mathcal{M}(\prec_{\mathcal{R}}) \models_{\mathit{\Psi},\pi} K(\mathbf{x}, b(\mathbf{x}))$. If $\mathcal{M}(\prec_{\mathcal{R}}) \models_{\mathit{\Psi},\alpha} \neg A(\alpha)$ then $\pi(x) = 1$; so $\mathcal{M}(\prec_{\mathcal{R}})$ satisfies RLS3 for $\neg A(\alpha)$ also.

Lemma 2. If $\alpha \in \mathcal{RLS}$, then (i) $w * \alpha \in \mathcal{RLS}$ for every finite sequence code w, (ii) $\alpha \circ \varphi \in \mathcal{RLS}$ for every injection $\varphi \in \mathcal{R}$ the characteristic function of whose range is also in $\mathcal{R}$, and (iii) if $\pi \in \mathcal{R}$ is a predictor and $n \in \omega$ then π correctly predicts α at some $m \geq n$. Hence any sequence obtained by changing or omitting finitely many values of any $\alpha \in \mathcal{RLS}$ is also in $\mathcal{RLS}$; if $[\alpha_1, \dots, \alpha_k] \in \mathcal{RLS}$ then $\alpha_i \in \mathcal{RLS}$ for each $i \leq k$; and every predictor $\pi \in \mathcal{R}$ correctly predicts every $\alpha \in \mathcal{RLS}$ infinitely many times.

Proofs. Assume $\alpha \in \mathcal{RLS}$.

(i) Since $\mathcal{M}(\prec_{\mathcal{R}})$ satisfies $\mathrm{AC}_{00}^R!$, for each finite sequence code w and each predictor $\pi \in \mathcal{R}$ there is a predictor $\rho \in \mathcal{R}$ such that $\rho(y) = \pi(w * y)$ for each finite sequence code y, and ρ correctly predicts α at some n, so π correctly predicts $w * \alpha$ at $lh(w) + n$.

[4]In general, "$\mathcal{M}(\prec_{\mathcal{R}})$ satisfies the axiom" means that $\mathcal{M}(\prec_{\mathcal{R}})$ satisfies the universal closure of the axiom; similarly for axiom schemas such as W4.

[5]In general, "$\alpha \in \mathcal{RLS}$" abbreviates "$\mathcal{M}(\prec_{\mathcal{R}}) \models_\alpha RLS(\alpha)$."

(ii) Let $\varphi, \psi, \pi \in \mathcal{R}$ where φ is an injection, ψ is the characteristic function of its range, and π is a predictor. The problem is to show that π correctly predicts $\alpha \circ \varphi$ somewhere. If y codes a finite sequence, let $y \circ \varphi$ be the sequence code of length $n = \max\{i : \varphi(i) < lh(y)\} + 1$ such that for $i < n$: $(y \circ \varphi)_i = (y)_{\varphi(i)}$ if $\varphi(i) < lh(y)$, and $(y \circ \varphi)_i = 1$ otherwise. Let $z = (y \circ \varphi) * \pi(y \circ \varphi)$ and define $\rho(y)$ to be the sequence code of length $m = \max\{\varphi(i) : i < lh(z)\} + 1 \dot{-} lh(y)$ such that for $j < m$: $(\rho(y))_j = (z)_k$ where $\varphi(k) = j + lh(y)$ if there is such a k, otherwise $(\rho(y))_j = 1$. If y is not a sequence code set $\rho(y) = 0$.

Then ρ is a predictor and $\rho \in \mathcal{R}$, so ρ correctly predicts α somewhere, so $\alpha \in y * \rho(y)$ for some finite sequence code y. To show: $\alpha \circ \varphi \in (y \circ \varphi) * \pi(y \circ \varphi) = z$ for this y. With n, m as above, suppose $i < lh(z)$. Then either $\varphi(i) < lh(y)$, so $(\alpha \circ \varphi)(i) + 1 = \alpha(\varphi(i)) + 1 = (y)_{\varphi(i)} = (y \circ \varphi)_i = (z)_i$; or else $\varphi(i) = lh(y) + j$ where $j < m = lh(\rho(y))$, so $lh(y) \leq \varphi(i) < lh(y * \rho(y))$, so $(\alpha \circ \varphi)(i) + 1 = \alpha(\varphi(i)) + 1 = (y * \rho(y))_{\varphi(i)} = (\rho(y))_j = (z)_i$. So $z = \overline{\alpha \circ \varphi}(lh(z))$, so π correctly predicts $\alpha \circ \varphi$ at n.

(iii) If π is a lawlike predictor and $n \in \omega$, define $\rho(y) = \pi(\overline{\alpha}(n) * y)$ for all finite sequence codes y, and $\rho(y) = 0$ otherwise. Then ρ is a lawlike predictor, so by (ii) (just established) ρ correctly predicts $\lambda x.\alpha(n + x)$ at some k, so $\alpha \in \overline{\alpha}(n + k) * \pi(\overline{\alpha}(n + k))$, so π correctly predicts α at $m = n + k$.

4.3. **Outline of the proof of Theorem 1.** The theorem can be restated as follows: Assume $\mathcal{R}$ is countable. Then

(a) The structure $\mathcal{M}(\prec_\mathcal{R})$ is a classical model of **RLS(≺)** with $\mathcal{R}$ as the class of lawlike sequences, and otherwise standard.

(b) The collection $\mathcal{RLS} = \{\alpha \in \omega^\omega \mid \mathcal{M}(\prec_\mathcal{R}) \models_\alpha RLS(\alpha)\}$ is disjoint from $\mathcal{R}$ and is Baire comeager in ω^ω, with classical measure 0.

Part of (a) has already been proved. For (b), the fact that $\mathcal{RLS}$ is disjoint from $\mathcal{R}$ is obvious, since if $\pi \in \mathcal{R}$ then $\sigma = \lambda x.\langle \pi(lh(x)) + 3\rangle \in \mathcal{R}$ is a lawlike predictor which never correctly predicts π. To prove that $\mathcal{M}(\prec_\mathcal{R})$ satisfies RLS1-RLS4, and to finish the proof of (b), an enumeration of $\mathcal{R}$ is needed.

Since $\mathcal{R}$ is an ordinal-definable subset of ω^ω and $\prec_\mathcal{R}$ is an ordinal-definable wellordering of $\mathcal{R}$, by [8] we may consistently assume that η_0, and hence $\mathcal{R}$, is countable. For the rest of this section, assume that $\Gamma : \omega \to \mathcal{R}$ is a bijection and $\chi : \omega \times \omega \to \{0, 1\}$ codes a wellordering of type η_0 such that for all $n, m \in \omega$:

$$\Gamma(n) \prec_\mathcal{R} \Gamma(m) \Leftrightarrow \chi(n, m) = 1.$$

To show that $\mathcal{M}(\prec_\mathcal{R})$ satisfies RLS1, define sequences $\{w_n\}_{n\in\omega}$, $\{x_n\}_{n\in\omega}$ as follows:

$$\begin{aligned} w_0 &= \langle\,\rangle = 1, \\ w_{n+1} &= \begin{cases} w_n * ((\Gamma(n))(w_n)) & \text{if } (\Gamma(n))(w_n) \text{ codes a finite sequence,} \\ w_n & \text{otherwise.} \end{cases} \\ x_n &= lh(w_n). \end{aligned}$$

Since $\mathcal{R}$ contains infinitely many predictors, $\{x_n\}_n$ is cofinal in ω so there is a unique $\alpha \in \omega^\omega$ such that $\overline{\alpha}(x_n) = w_n$ for each n. Evidently this $\alpha \in \mathcal{RLS}$.

To show that $\mathcal{M}(\prec_\mathcal{R})$ satisfies RLS2, suppose that $\alpha \in \mathcal{RLS}$ and w codes a finite sequence, and define $\overline{\beta}(lh(w)) = w$. Given any predictor $\pi \in \mathcal{R}$ and any prior choice

$\overline{\beta}(n)$ with $n \geq lh(w)$, there are predictors $\varphi, \psi \in \mathcal{R}$ so that for finite sequence codes y:

$$\varphi(y) = \pi(\overline{[y * \lambda x.0, \overline{\beta}(n) * \lambda x.0]}(2 \cdot lh(y))),$$
$$\psi(y) = \Pi_{2i<lh(\varphi(y))} p_i^{(\varphi(y))_{2i}}.$$

By Lemma 2 there is a least $m > n$ such that $\alpha \in \overline{\alpha}(m) * \psi(\overline{\alpha}(m))$, and then $\varphi(\overline{\alpha}(m)) = \pi(\overline{[\alpha, \overline{\beta}(n) * \lambda x.0]}(2m))$. If $\overline{\beta}(n)$ is extended by setting $\beta(i) = 0$ for $n \leq i < m$ and $\beta(i) = (\varphi(\overline{\alpha}(m)))_{2i+1} \dot{-} 1$ for $2i+1 < lh(\varphi(\overline{\alpha}(m)))$, then π will correctly predict $[\alpha, \beta]$ at $2m$. Treating each of the countably many predictors $\pi \in \mathcal{R}$ in turn in this manner gradually produces a β such that $[\alpha, \beta] \in \mathcal{RLS}$.

The proof that $\mathcal{M}(\prec_{\mathcal{R}})$ satisfies RLS3 is by induction on the logical form of $A(\alpha)$. The cases $s(\alpha) = t(\alpha)$ and $u \prec v$ (u, v R-functors) are immediate. The case $u(\alpha) \prec v(\alpha)$ uses W5, so is treated after the inductive cases for $\forall a$, $\forall x$ and &. In addition to establishing the inductive case for $\neg$, since $\mathcal{M}(\prec_{\mathcal{R}})$ is a classical model Lemma 1 gives the inductive cases for $\vee$, $\rightarrow$ and $\forall$ from those for & and $\exists$, simplifying the proof.

The proof that $\mathcal{M}(\prec_{\mathcal{R}})$ satisfies RLS4 uses the fact that $\mathcal{M}(\prec_{\mathcal{R}})$ satisfies RLS3, AC^R_{01} and RLEM (cf. [9]). This completes the proof of Theorem 1(a). The remaining parts of the proof of (b) are straightforward (cf. [9] and [10]).

5. The Γ-realizability interpretation

5.1. **Definitions.** Assume $\mathcal{R}$ is countable, with Γ, χ as in the proof of Theorem 1, and let $\Psi = x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_k, a_1, \ldots, a_m$ be a list of distinct variables. Then a Γ-*interpretation* $\mathit{\Psi}$ of Ψ is any choice of n numbers, k elements of ω^ω and m numbers $r_1, \ldots, r_m$; and $\Gamma[\mathit{\Psi}]$ is the corresponding list of n numbers, k elements of ω^ω and m elements $\Gamma(r_1), \ldots, \Gamma(r_m)$ of $\mathcal{R}$.

With the same assumptions on Γ and χ, the Γ-realizability interpretation of $\mathcal{L}(\prec)$ is defined as follows. For $\pi \in \omega^\omega$, E a formula of $\mathcal{L}(\prec)$ with at most the distinct variables Ψ free, and $\mathit{\Psi}$ a Γ-interpretation of Ψ, define π Γ-*realizes*-$\mathit{\Psi}$ E by induction on the logical form of E using Kleene's coding of recursive partial functionals:[6]

(1) π Γ-realizes-$\mathit{\Psi}$ a prime formula P, if $\mathcal{M}(\prec_{\mathcal{R}}) \models_{\Gamma[\mathit{\Psi}]} P$.
(2) π Γ-realizes-$\mathit{\Psi}$ A & B, if $(\pi)_0$ Γ-realizes-$\mathit{\Psi}$ A and $(\pi)_1$ Γ-realizes-$\mathit{\Psi}$ B.
(3) π Γ-realizes-$\mathit{\Psi}$ $A \vee B$, if $(\pi(0))_0 = 0$ and $(\pi)_1$ Γ-realizes-$\mathit{\Psi}$ A, or $(\pi(0))_0 \neq 0$ and $(\pi)_1$ Γ-realizes-$\mathit{\Psi}$ B.
(4) π Γ-realizes-$\mathit{\Psi}$ $A \rightarrow B$, if, if σ Γ-realizes-$\mathit{\Psi}$ A, then $\{\pi\}[\sigma]$ Γ-realizes-$\mathit{\Psi}$ B.
(5) π Γ-realizes-$\mathit{\Psi}$ $\neg A$, if π Γ-realizes-$\mathit{\Psi}$ $A \rightarrow 1 = 0$.
(6) π Γ-realizes-$\mathit{\Psi}$ $\forall x A(x)$, if for each $x \in \omega$, $\{\pi\}[x]$ Γ-realizes-$\mathit{\Psi}, x$ $A(x)$.
(7) π Γ-realizes-$\mathit{\Psi}$ $\exists x A(x)$, if $(\pi)_1$ Γ-realizes-$\mathit{\Psi}, (\pi(0))_0$ $A(x)$.
(8) π Γ-realizes-$\mathit{\Psi}$ $\forall a A(a)$, if for each $r \in \omega$, $\{\pi\}[r]$ Γ-realizes-$\mathit{\Psi}, r$ $A(a)$.
(9) π Γ-realizes-$\mathit{\Psi}$ $\exists a A(a)$, if $(\pi)_1$ Γ-realizes-$\mathit{\Psi}, (\pi(0))_0$ $A(a)$.
(10) π Γ-realizes-$\mathit{\Psi}$ $\forall \alpha A(\alpha)$, if for each $\alpha \in \omega^\omega$, $\{\pi\}[\alpha]$ Γ-realizes-$\mathit{\Psi}, \alpha$ $A(\alpha)$.
(11) π Γ-realizes-$\mathit{\Psi}$ $\exists \alpha A(\alpha)$, if $(\pi)_1$ Γ-realizes-$\mathit{\Psi}, \{(\pi)_0\}$ $A(\alpha)$.

Here e.g. "$\{\pi\}[\alpha]$ Γ-realizes-$\mathit{\Psi}, \alpha$ $A(\alpha)$" abbreviates "$\{\pi\}[\alpha]$ is completely defined and Γ-realizes-$\mathit{\Psi}, \alpha$ $A(\alpha)$."

[6] $\{\pi\}[\alpha] \simeq \lambda x.\ \pi(\langle x+1 \rangle * \overline{\alpha}(\mu z\ \pi(\langle x+1 \rangle * \overline{\alpha}(z)) > 0)) \dot{-} 1$ and $\{\pi\}[x] \simeq \{\pi\}[\lambda y.\ x]$.

5.2. **Outline of the proof of Theorem 2.** The theorem can be restated as follows:

Assume $\mathcal{R}$ is countable, with Γ, χ as in the proof of Theorem 1. Then to each theorem E of **FIRM(≺)** with at most the distinct variables $\Psi = \Psi_0, \Psi_1, \Psi_2$ free, there is a function $\varphi[\mathit{\Psi}]$ which is continuous in $\mathit{\Psi}$ and Γ-realizes-$\mathit{\Psi}$ E for each Γ-interpretation $\mathit{\Psi}$ of Ψ. Since $0 = 1$ is not Γ-realizable, **FIRM(≺)** and **FIRM** are consistent.

The proof is by induction on a derivation of E, using three lemmas. Assume Γ, χ are as in the proof of Theorem 1, and let "E is true-$\Gamma[\mathit{\Psi}]$" abbreviate "$\mathcal{M}(\prec_{\mathcal{R}}) \models_{\Gamma[\mathit{\Psi}]} E$."

Lemma 3. To each list Ψ of distinct *number and lawlike sequence* variables and each restricted R-formula $A(x, y)$ containing free at most Ψ, x, y where $x, y, a \notin \Psi$, there is a partial function $\xi_A[\mathit{\Psi}]$ so that for each Γ-interpretation $\mathit{\Psi}$ of Ψ: If $\forall x \exists ! y A(x, y)$ is true-$\Gamma[\mathit{\Psi}]$ then $\xi_A[\mathit{\Psi}]$ is defined and $\forall x A(x, a(x))$ is true-$\Gamma[\mathit{\Psi}, \xi_A[\mathit{\Psi}]]$.

Lemma 4. To each list $\Psi = x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_k, a_1, \ldots, a_m$ of distinct variables and each almost negative formula E of $\mathcal{L}[\prec]$ containing free only Ψ there is a function $\varepsilon_E[\mathit{\Psi}] = \lambda t.\varepsilon_E(\mathit{\Psi}, t)$ partial recursive in Γ so that for each Γ-interpretation $\mathit{\Psi}$ of Ψ:

(i) If E is Γ-realized-$\mathit{\Psi}$ then E is true-$\Gamma[\mathit{\Psi}]$, and
(ii) E is true-$\Gamma[\mathit{\Psi}]$ if and only if $\varepsilon_E[\mathit{\Psi}]$ is completely defined and Γ-realizes-$\mathit{\Psi}$ E.

Lemma 5. To each list $\Psi = x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_k, a_1, \ldots, a_m$ and each restricted formula E of $\mathcal{L}(\prec)$ containing free at most Ψ, there is a continuous partial function $\zeta_E[\mathit{\Psi}]$ such that for each Γ-interpretation $\mathit{\Psi}$ of Ψ with $[\alpha_1, \ldots, \alpha_k] \in \mathcal{R}LS$:

(i) If E is Γ-realized-$\mathit{\Psi}$ then E is true-$\Gamma[\mathit{\Psi}]$, and
(ii) E is true-$\Gamma[\mathit{\Psi}]$ if and only if $\zeta_E[\mathit{\Psi}]$ is completely defined and Γ-realizes-$\mathit{\Psi}$ E.

The proof of Lemma 5 uses a sublemma: For each list $\Psi = \Psi_0, \Psi_1, \Psi_2$ of distinct variables where $\Psi_1 = \alpha_0, \ldots, \alpha_{k-1}$ and Ψ_0, Ψ_2 are number and lawlike sequence variables respectively, and each restricted formula $E(\Psi)$ with no other variables free, by Theorem 1 there is a partial continuous functional $\{\tau_E[\mathit{\Psi}_0, \mathit{\Psi}_2]\}(\alpha)$ such that if $\mathit{\Psi}$ is a Γ-interpretation of Ψ with $\mathit{\Psi}_1 = \alpha_0, \ldots, \alpha_{k-1}$ where $\alpha = [\alpha_0, \ldots, \alpha_{k-1}] \in \mathcal{R}LS$, then $\{\tau_E[\mathit{\Psi}_0, \mathit{\Psi}_2]\}(\alpha)$ is defined and equal to 0 if E is true-$\Gamma[\mathit{\Psi}]$, or 1 if $\neg E$ is true-$\Gamma[\mathit{\Psi}]$.

To complete the proof of the theorem, observe that Γ-realizability extends the function-realizability interpretation of [6] to $\mathcal{L}(\prec)$, so if E is an axiom of **FIM** or an axiom schema of **FIM** + GC_{11} extended to $\mathcal{L}(\prec)$ then E is Γ-realized-$\mathit{\Psi}$ (essentially as in Theorem 10 of [6]) by a $\varphi[\mathit{\Psi}]$ primitive recursive in Γ. For the additional axioms, schemas and rules of **FIRM(≺)** use Theorem 1 with Lemmas 3-5. More details are in [11].

6. Epilogue

The results show that Brouwer's continuum can consistently contain all the lawlike and lawless sequences. Considered separately, these classes are distinguished by the axioms they satisfy:

(a) Brouwer's arbitrary choice sequences satisfy the bar theorem, countable choice and continuous choice. Their logic is intuitionistic. Equality of choice sequences is not decidable.
(b) The lawlike sequences satisfy the bar theorem and countable choice, but not continuous choice. Their logic is classical. Equality of lawlike sequences is decidable.

(c) The lawless sequences satisfy (restricted) open data and restricted continuous choice, but not the restricted bar theorem. The restricted logic of independent lawless sequences is classical, but equality of lawless sequences is not decidable.

Kleene observed in [6] that when considered separately, the recursive sequences do not satisfy the fan theorem, the arithmetical sequences satisfy the fan theorem, but even the hyperarithmetical sequences do not satisfy the bar theorem. In $\mathcal{M}(\prec_{\mathcal{R}})$ all analytic sequences are lawlike, but the three-sorted system $\mathbf{FIRM}(\prec)$ does not obviously entail $\forall x \exists ! y A(x, y) \to \exists b \forall x A(a, b(x))$ for all formulas $A(x, y)$ of $\mathcal{L}_{\mathbf{FIM}}$ with only x, y free. This may be an open question.

Lawless and *random* are orthogonal concepts. A random sequence of natural numbers should possess certain definable regularity properties (e.g. the percentage of even numbers in its nth initial segment should approach .50 as n increases), while a lawless sequence should possess none. Any regularity property definable in $\mathcal{L}$ by a restricted formula can be defeated by a suitable lawlike predictor.

The model considered here has an explicit forcing characterization (cf. [10]). Under the assumption that there are only countably many lawlike sequences, α is lawless if and only if $\overline{\alpha} = \{\overline{\alpha}(n) : n \in \omega\}$ is generic with respect to the class of all dense subsets of $\omega^{<\omega}$ which are classically definable in $\mathcal{L}(\prec)$ by restricted formulas with lawlike parameters.

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