"At Most One" Constructively

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Question: What does "there is at most one $x \in X$ satisfying $P(x)$" mean constructively if X is $\mathbb{N}$, or $2^{\mathbb{N}}$, or $\mathbb{N}^{\mathbb{N}}$, or a canonical set $\mathbb{R}$ of real-number generators, or any constructive measure space?

Context: X comes equipped with an equality relation $=$ with respect to which $P(x)$ is assumed to be extensional.

Equality on $\mathbb{N}$ is decidable, in fact $\mathbb{N}$ has a primitive recursive linear ordering $<$ satisfying $\quad x = y \ \vee \ x < y \ \vee \ y < x$.

Equality on $\mathbb{N}^{\mathbb{N}}$ and $2^{\mathbb{N}}$ is defined in terms of equality on $\mathbb{N}$: $\alpha = \beta \equiv \forall x \alpha(x) = \beta(x)$ and is *stable*, i.e. $\neg\neg(\alpha = \beta) \to \alpha = \beta$, but *not decidable* from the constructive viewpoint.

Equality on $\mathbb{R}$ is also Π^0_1-definable, stable, but not decidable.

$\mathbb{N}^{\mathbb{N}}$, $2^{\mathbb{N}}$ and $\mathbb{R}$ have Σ^0_1-definable linear orderings which are not decidable, and not stable unless Markov's Principle is accepted.

Definition. If $x, y \in X$ where X is linearly ordered by $<$, then x is *apart from* y if and only if $x < y$ or $y < x$.

If X is $\mathbb{N}$ then apartness coincides with inequality, but generally apartness is stronger. The negation of apartness coincides with equality because $\neg\Sigma^0_1 = \Pi^0_1$ constructively as well as classically.

"At most one x in X satisfies $P(x)$" could be expressed constructively by either of

1. For all $x, y \in X$: if $P(x)$ & $P(y)$ then $x = y$.
2. For all $x, y \in X$: if x is apart from y then $\neg P(x) \vee \neg P(y)$.

Constructively $2 \Rightarrow 1$ but $1 \not\Rightarrow 2$. Two other variants are constructively equivalent to 1, which is straightforward, but 2 is also interesting. In the case $P \leftrightarrow \neg Q$, a stronger version of 2 is

2'. For all $x, y \in X$: if x is apart from y then $Q(x) \vee Q(y)$.

Brouwer's Constructive (Intuitionistic) Perspective

The early constructivist L. E. J. Brouwer rejected arbitrary use of the law of excluded middle $A \vee \neg A$ but accepted

- the logical law $\neg\neg\neg A \to \neg A$, where $\neg A \equiv (A \to 0 = 1)$
- full induction on the natural numbers $\mathbb{N}$
- countable and dependent choice
- full (monotone) bar induction on the "universal spread" $\mathbb{N}^{\mathbb{N}}$, whose "points" include (but are not restricted to)
  - "free choice" sequences generated one choice at a time, and
  - "lawlike" sequences, determined in advance
- continuous choice (so every total function from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$ is continuous in the finite-initial-segment topology)
- real numbers represented by Cauchy sequences of rationals, or by nested sequences of intervals with rational endpoints.

Brouwer had no problem using *reductio ad absurdum* to derive negative conclusions but his proofs of existential assertions always provided (constructive approximations to) witnesses.

Brouwer worked informally, but Heyting [1930] published formal systems for intuitionistic propositional logic, first-order predicate logic with equality, number theory and parts of analysis.

Kolmogorov [1932] interpreted intuitionistic logic, e.g.:

- An implication $A \to B$ expresses the problem of finding a general method for reducing the problem B to the problem A.
- A universal statement $\forall x A(x)$ expresses the problem of finding a general method for solving the problem $A(x)$.
- An existential statement $\exists x A(x)$ expresses the problem of finding a witness.

Heyting used proofs instead of problems.

Artemov and his colleagues have perfected Heyting's version of this "B-H-K" explication of intuitionistic predicate logic.

Formalization and consistency: Kleene-Vesley [1965]

Kleene and Vesley [1965] proposed an intuitionistic formal system **FIM**, based on extensional two-sorted intuitionistic number theory $\mathbf{IA}_1$, with axioms for the primitive recursive function(al) constants and axiom schemas for countable and continuous choice and the "bar theorem." Kleene proved that **FIM** is consistent relative to its classically correct subsystem **B** by interpreting B-H-K recursively, and Vesley formalized part of Brouwer's real analysis.

Kleene [1969] developed the theory of recursive partial functionals and function-realizability in a proper subsystem $\mathbf{M} = \mathbf{IA}_1 + \mathrm{AC}_{00}!$ of **B**, where $\mathrm{AC}_{00}!$ is the axiom schema of "unique choice":

$$\forall x \exists ! y A(x, y) \rightarrow \exists \alpha \forall x A(x, \alpha(x)).$$

Here $\exists ! y B(y)$ abbreviates $\exists y B(y)$ & $\forall y \forall z (B(y)$ & $B(z) \rightarrow y = z)$ expressing "at most one" by option 1 in this context.

"There is exactly one" ($\exists!$)

Proposition 1. The notations $\exists! x B(x)$ and $\exists! \alpha B(\alpha)$ are neutral with respect to expressing "at most one" by option 1 or 2.

Proof. Intuitionistic two-sorted arithmetic $\mathbf{IA}_1$ proves

$\exists y B(y) \;\&\; \forall y \forall z (B(y) \;\&\; B(z) \to y = z) \to \forall y (B(y) \vee \neg B(y))$

so over $\mathbf{IA}_1$ it doesn't matter whether $\exists! y B(y)$ denotes

(1) $\exists y B(y) \;\&\; \forall y \forall z (B(y) \;\&\; B(z) \to y = z)$ or

(2) $\exists y B(y) \;\&\; \forall y \forall z (y < z \vee z < y \to \neg B(y) \vee \neg B(z))$.

By a different argument, the following are equivalent over $\mathbf{IA}_1$, so it doesn't matter which of the two $\exists! \alpha B(\alpha)$ denotes:

(3) $\exists \alpha B(\alpha) \;\&\; \forall \beta \forall \gamma (B(\beta) \;\&\; B(\gamma) \to \forall x \beta(x) = \gamma(x))$,

(4) $\exists \alpha B(\alpha) \;\&\; \forall \beta \forall \gamma (\exists x \beta(x) \neq \gamma(x) \to \neg B(\beta) \vee \neg B(\gamma))$.

Caveat. If $B \leftrightarrow \neg C$ where C is not stable under double negation, then option $2'$ would be stronger in this context.

Decidable = detachable = $\exists$ a characteristic function

Proposition 2. (with G Vafeiadou) $\mathbf{M} = \mathbf{IA}_1 + \mathrm{AC}_{00}!$ proves

- $\mathrm{AC}_{01}!$: $\quad \forall x \exists! \alpha A(x, \alpha) \rightarrow \exists \beta \forall x A(x, (\beta)^x)$
  where $(\beta)^x$ is $\lambda y \beta(\langle x, y \rangle)$ (the xth section of β).
- the schema $\mathrm{CF_d}$: *every decidable predicate of numbers defines a detachable species, which has a characteristic function*:
  $\forall x(A(x) \vee \neg A(x)) \rightarrow \exists \rho \forall x(\rho(x) \leq 1 \;\&\; (\rho(x) = 0 \leftrightarrow A(x)))$.
- $\mathrm{QF\text{-}AC}_{00}$: $\quad \forall \mathrm{x} \exists \mathrm{y} \mathrm{A}(\mathrm{x}, \mathrm{y}) \rightarrow \exists \alpha \forall \mathrm{x} \mathrm{A}(\mathrm{x}, \alpha(\mathrm{x}))$
  for A quantifier-free or with bounded number quantifiers.

Note that $\mathbf{IA}_1 + \mathrm{CF_d} \nvdash \mathrm{QF\text{-}AC}_{00}$ and $\mathbf{IA}_1 + \mathrm{QF\text{-}AC}_{00} \nvdash \mathrm{CF_d}$.

Theorem 3. (Garyfallia Vafeiadou [2012]):

- $\mathbf{M} = \mathbf{IA}_1 + \mathrm{CF_d} + \mathrm{QF\text{-}AC}_{00}$.
- $\mathbf{M}$ is definitionally equivalent to $\mathbf{EL} + \mathrm{CF_d}$, where $\mathbf{EL}$ is the system of Troelstra [1973] with $\mathrm{QF\text{-}AC}_{00}$; so $\mathbf{EL} \nvdash \mathrm{CF_d}$.

If there is at most one, is there one?

A different (and stronger) version of unique choice is $\mathrm{AC}_{00}!!$:

$$\begin{aligned}&\forall x \exists y A(x,y)\\ &\qquad \& \ \forall\alpha\forall\beta[\forall x A(x,\alpha(x)) \ \& \ \forall x A(x,\beta(x)) \rightarrow \alpha = \beta]\\ &\qquad\qquad \rightarrow \exists\alpha\forall x A(x,\alpha(x)).\end{aligned}$$

This version has an interesting relationship with $\mathrm{AC}_{00}!$ and the axiom of countable choice AC_{00}: $\forall x \exists y A(x,y) \rightarrow \exists\alpha\forall x A(x,\alpha(x))$.

Theorem 4. Over intuitionistic two-sorted arithmetic $\mathbf{IA}_1$,

$$\mathrm{AC}_{00} \Rightarrow \mathrm{AC}_{00}!! \Leftrightarrow \mathrm{AC}_{00}! + \neg\neg\, \mathrm{AC}_{00}.$$

Corollary 5. Over $\mathbf{M}$ $(= \mathbf{IA}_1 + \mathrm{AC}_{00}!)$, $\quad \mathrm{AC}_{00}!! \Leftrightarrow \neg\neg\, \mathrm{AC}_{00}$.

It doesn't matter whether "at most one" in the statement of $\mathrm{AC}_{00}!!$ is expressed by option 1 or 2.

Bishop's Cautious Constructive Perspective

Bishop [1967] based his cautious constructivism on a strictly positive, computational version of the B-H-K explication. He accepted countable and dependent choice but remained neutral on Brouwer's bar theorem and continuous choice principle.

Bridges, Richman, Ishihara and many others continue Bishop's work. Recent studies of unique existence by Bishop constructivists interpret "at most one" by option $2'$ or a potentially stronger option 3:

"In order to introduce the idea of uniqueness without *a priori* existence," Bridges [2005] advocates expressing "there is at most one element of X with the property P" by

$$\forall x, y \in X(x \neq y \rightarrow P'(x) \vee P'(y))$$

where P' is "some strong form of negation of P" and "$\neq$" denotes apartness.

A reasonable basis for constructive reverse analysis?

Until recently, Bishop's followers have worked informally. But in order to prove sharp results in constructive reverse mathematics CRM, some kind of formalization (or at least axiomatization) seems essential.

Aczel and Rathjen's precise treatment of constructive and intuitionistic set theory provides a basis for formalizing all of CRM.

But constructive analysis can be expressed simply in a two- or at most three-sorted language of numbers and functions. Most results in constructive reverse analysis CRA can be understood over $\mathbf{IA}_1$ or $\mathbf{EL}$ ($\approx \mathbf{IA}_1 + \text{QF-AC}_{00}$) or $\mathbf{M}$.

For the rest of this talk, by *"constructively equivalent"* we mean "equivalent over $\mathbf{M}$."

Weak König's Lemma with uniqueness (option 2′)

"Weak König's Lemma" WKL is König's Lemma KL for *detachable* subtrees of $2^{\mathbb{N}}$. WKL is constructively equivalent to

$$\forall y \exists \alpha \in 2^{\mathbb{N}} \forall x \leq y \rho(\overline{\alpha}(x)) = 0 \rightarrow \exists \alpha \in 2^{\mathbb{N}} \forall x \rho(\overline{\alpha}(x)) = 0.$$

Adding a strong effective uniqueness hypothesis gives WKL!:

$$\begin{aligned} &\forall y \exists \alpha \in 2^{\mathbb{N}} \forall x \leq y \rho(\overline{\alpha}(x)) = 0 \\ &\& \ \forall \alpha \in 2^{\mathbb{N}} \forall \beta \in 2^{\mathbb{N}} [\exists x \alpha(x) \neq \beta(x) \rightarrow \exists x \rho(\overline{\alpha}(x)) \neq 0 \vee \exists x \rho(\overline{\beta}(x)) \neq 0] \\ &\qquad \rightarrow \exists \alpha \in 2^{\mathbb{N}} \forall x \rho(\overline{\alpha}(x)) = 0. \end{aligned}$$

Theorem 6. (Ishihara, J. Berger, Schwichtenberg, all [2005]) WKL! is constructively equivalent to Brouwer's fan theorem $\mathrm{FT_d}$:

$$\forall \alpha \in 2^{\mathbb{N}} \exists x \rho(\overline{\alpha}(x)) = 0 \rightarrow \exists y \forall \alpha \in 2^{\mathbb{N}} \exists x \leq y \rho(\overline{\alpha}(x)) = 0.$$

Weak König's Lemma with uniqueness (option 1)

Weakening the uniqueness hypothesis in WKL! gives WKL!!:

$$
\begin{aligned}
&\forall y \exists \alpha \in 2^{\mathbb{N}} \forall x \le y \rho(\overline{\alpha}(x)) = 0 \\
&\& \ \forall \alpha \in 2^{\mathbb{N}} \forall \beta \in 2^{\mathbb{N}} [\forall x \rho(\overline{\alpha}(x)) = 0 \ \& \ \forall x \rho(\overline{\beta}(x)) = 0 \to \alpha = \beta] \\
&\qquad\qquad\qquad\qquad\qquad\qquad \to \exists \alpha \in 2^{\mathbb{N}} \forall x \rho(\overline{\alpha}(x)) = 0.
\end{aligned}
$$

Proposition 7. Constructively, WKL $\Rightarrow$ WKL!! $\Rightarrow$ WKL!.

Theorem 8. Constructively, $\mathrm{FT_d} \not\Rightarrow$ WKL!! (so WKL! $\not\Rightarrow$ WKL!!) and WKL!! $\not\Rightarrow$ WKL.

Outline of Proof. Decompose WKL!! into a logical principle $\mathrm{MP}^{\vee}$ and a mathematical one $\neg\neg$ WKL, following the example of Ishihara's decomposition [2005] of WKL and Berger's [2009] of WKL!. Establish $\not\Rightarrow$s using realizability arguments.

In a little more detail: Constructively,

$$\mathsf{WKL!!} \Leftrightarrow \mathsf{MP}^{\vee} + \neg\neg\, \mathsf{WKL}$$

where $\mathsf{MP}^{\vee}$ is

$\neg\neg\exists x(\alpha(x) \neq 0 \vee \beta(x) \neq 0) \to \neg\neg\exists x\alpha(x) \neq 0 \vee \neg\neg\exists x\beta(x) \neq 0.$

To prove that WKL! $\not\Rightarrow$ WKL!! $\not\Rightarrow$ WKL, recall that $\mathsf{FT}_{\mathrm{d}} \Leftrightarrow$ WKL! and observe:

- FT_{d} and WKL!! are Kleene recursive function-realizable.
- FT_{d} is also ^{G}realizable (JRM [1971]), but $\mathsf{MP}^{\vee}$ is not; so WKL!! is not ^{G}realizable.
- WKL is not even Kleene recursive function-realizable, by Kleene's example of a recursive subtree of the binary tree which has (recursively) arbitrarily long finite branches but no recursive infinite branch.

König's Lemma with and without uniqueness

The general form of König's Lemma KL for the binary fan is

$$\forall y \exists \alpha \in 2^{\mathbb{N}} \forall x \leq y R(\overline{\alpha}(x)) \to \exists \alpha \in 2^{\mathbb{N}} \forall x R(\overline{\alpha}(x)).$$

KL!! is like KL but with the additional hypothesis

$$\forall \alpha \in 2^{\mathbb{N}} \forall \beta \in 2^{\mathbb{N}} [\forall x R(\overline{\alpha}(x)) \ \& \ \forall x R(\overline{\beta}(x)) \to \alpha = \beta]$$

expressing "at most one" by option 1.

KL! is like KL but with the additional hypothesis

$$\forall \alpha \in 2^{\mathbb{N}} \forall \beta \in 2^{\mathbb{N}} [\exists x \alpha(x) \neq \beta(x) \to \exists x \neg R(\overline{\alpha}(x)) \vee \exists x \neg R(\overline{\beta}(x))]$$

expressing "at most one" by a version of option 3.

Proposition 9. Constructively, KL $\Rightarrow$ KL!! $\Rightarrow$ KL!.

Bishop constructivists have studied the results of strengthening FT_{d} by weakening the decidability requirement to a Π^0_1 monotone condition of one kind or another. Very recently Lubarsky and Diener succeeded in separating two of these versions from each other and from FT_{d}, using Kripke models.

König's Lemma is a classical contrapositive of the Fan Theorem, so it is natural to consider the versions $\mathsf{KL}(\Sigma^0_1)$ and $\mathsf{KL!!}(\Sigma^0_1)$ of KL and KL!! when the predicate R is Σ^0_1. E.g., $\mathsf{KL}(\Sigma^0_1)$ is

$$\forall y \exists \alpha \in 2^{\mathbb{N}} \forall x \leq y \exists z \sigma(\langle \overline{\alpha}(x), z \rangle) = 0 \\ \rightarrow \exists \alpha \in 2^{\mathbb{N}} \forall x \exists z \sigma(\langle \overline{\alpha}(x), z \rangle) = 0.$$

Theorem 10. Constructively, $\mathsf{KL}(\Sigma^0_1) \Rightarrow \mathsf{KL!!}(\Sigma^0_1) \Rightarrow \mathsf{WKL}$.

Since $\mathbf{M} + \mathsf{WKL}$ already fails to satisfy the Church-Kleene Rule, maybe this is a good place to stop.

The Church-Kleene Rule

As an axiom schema, Church's thesis seems to be overly restrictive. As a rule, it is admissible in most constructive theories.

M has a decidable formula $T_1(e, x, y)$ numeralwise expressing "y is a Gödel number of a computation of $\{e\}(x)$," and a p-functor $U(y)$ numeralwise representing the result-extracting function.

Theorem. (Kleene [1967], [1969]): If **T** is **FIM**, **B** or **M** then

- If $\mathbf{T} \vdash \forall x \exists y A(x, y)$ where only x, y are free in $A(x, y)$, then there is a Gödel number e such that $\mathbf{M} \vdash \forall x \exists! y T_1(\mathbf{e}, x, y)$ and $\mathbf{T} \vdash \forall x \forall y [T_1(\mathbf{e}, x, y) \to A(x, U(y))]$.
- If $\vdash \exists \alpha A(\alpha)$ where only α is free in $A(\alpha)$, then there is an e such that $\mathbf{M} \vdash \forall x \exists! y T_1(\mathbf{e}, x, y)$ and $\mathbf{T} \vdash \forall \alpha [\forall x \forall y (T_1(\mathbf{e}, x, y) \to U(y) = \alpha(x)) \to A(\alpha)]$.

Remark. If **T** is **FIM** + MP + WKL!!, a similar result holds with **M** + MP in place of **M**.

Happy Birthday, Sergei!

Recent references:

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