ANOTHER UNIQUE WEAK KÖNIG'S LEMMA WKL!!

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Abstract. In [2] J. Berger and Ishihara proved, via a circle of informal implications involving countable choice, that Brouwer's Fan Theorem for detachable bars on the binary fan is equivalent in Bishop's sense to various principles including a version WKL! of Weak König's Lemma with a strong effective uniqueness hypothesis. Schwichtenberg [9] proved the equivalence directly and formalized his proof in Minlog. We verify that his result does not require countable choice, and derive a separation principle SP from the Fan Theorem, in a minimal intuitionistic system **M** of analysis with function comprehension.

In contrast, WKL!! comes from Weak König's lemma WKL by adding the hypothesis that any two infinite paths must agree. WKL!! is interderivable over **M** with the conjunction of a consequence of Markov's Principle and the double negation of WKL. This decomposition is in the spirit of Ishihara's [4] and J. Berger's [1]. Kleene's function realizability and the author's modified realizability establish that WKL!! is strictly weaker than WKL and strictly stronger than WKL!.

1. Preliminaries

As in [8] let $\mathbf{IA}_1$ be a two-sorted theory with number variables $\mathrm{a, b, c}, \ldots$ and choice sequence variables $\alpha, \beta, \gamma, \ldots$ (with or without subscripts); Kleene's finite list ([6], [5]) of constants for particular primitive recursive functions and functionals, with their defining axioms; Church's λ, parentheses denoting function application, and the axiom schema of λ-reduction; the single relation constant $=$ for number-theoretic equality, with the axioms of reflexivity, symmetry and transitivity; the open equality axiom $\mathrm{x} = \mathrm{y} \rightarrow \alpha(\mathrm{x}) = \alpha(\mathrm{y})$; mathematical induction for all formulas of the two-sorted language; and intuitionistic two-sorted logic. Let $\mathbf{M}$ result from $\mathbf{IA}_1$ by adding the function comprehension schema $\mathrm{AC}_{00}!$:

$$\forall \mathrm{x} \exists ! \mathrm{y} \mathrm{A}(\mathrm{x}, \mathrm{y}) \rightarrow \exists \alpha \forall \mathrm{x} \mathrm{A}(\mathrm{x}, \alpha(\mathrm{x})).[1]$$

This is the "minimal" theory of [5], which Kleene used to formalize the theory of recursive partial functionals and his function-realizability for intuitionistic analysis. Pairs (x, y) and finite sequences $\langle \mathrm{x}_0, \ldots, \mathrm{x}_\mathrm{k} \rangle$ of natural numbers are coded and decoded primitive recursively; if u, v code sequences then $\mathrm{u} * \mathrm{v}$ codes their concatenation, and $\mathrm{lh}(\mathrm{u})$ is the length of the sequence coded by u. Finite initial segments of α are denoted by $\overline{\alpha}(0) = \langle\,\rangle = 1$ (the empty sequence code) and $\overline{\alpha}(\mathrm{n}+1) = \langle \alpha(0) \ldots, \alpha(\mathrm{n}) \rangle$.

[1] In general, $\exists ! \mathrm{y} \mathrm{B}(\mathrm{y})$ abbreviates $\exists \mathrm{y} \mathrm{B}(\mathrm{y})$ & $\forall \mathrm{y} \forall \mathrm{z}(\mathrm{B}(\mathrm{y})$ & $\mathrm{B}(\mathrm{z}) \rightarrow \mathrm{y} = \mathrm{z})$. With intuitionistic logic, $\mathrm{AC}_{00}!$ is weaker than countable choice AC_{00} (without the "!").

G. Vafeiadou has shown that **M** is essentially equivalent to Troelstra's **EL** plus an axiom schema $\mathrm{CF_d}$ asserting that every decidable relation on the natural numbers has a characteristic function (cf. [8]). It seems natural to choose **M** as a basis for general studies of constructive entailment.

Proposed Terminology. Two schemas E and F will be called *constructively equivalent* if every instance of each is derivable in **M** from instances of the other. If every instance of F is derivable in **M** from instances of E, then E *constructively entails* F (and F *is a constructive consequence of* E).

The tree of all finite sequences of 0s and 1s is called *the binary fan.* Let 2^* abbreviate the set of all codes of finite binary sequences, and $2^{\mathbb{N}}$ the set of all infinite binary sequences, so $\mathrm{w} \in 2^* \leftrightarrow \exists \alpha \in 2^{\mathbb{N}}[\mathrm{w} = \overline{\alpha}(\mathrm{lh(w)})]$ and $\alpha \in 2^{\mathbb{N}} \leftrightarrow \forall \mathrm{x} \alpha(\mathrm{x}) \leq 1$.

In **M**, a classically and intuitionistically correct statement of Brouwer's Fan Theorem for detachable bars on the binary fan is $\mathrm{FT_d}$:

$$\forall \alpha \in 2^{\mathbb{N}} \exists \mathrm{x} \rho(\overline{\alpha}(\mathrm{x})) = 0 \rightarrow \exists \mathrm{y} \forall \alpha \in 2^{\mathbb{N}} \exists \mathrm{x} \leq \mathrm{y} \rho(\overline{\alpha}(\mathrm{x})) = 0.$$

Intuitionistic mathematics also accepts the full Fan Theorem FT:

$$\forall \alpha \in 2^{\mathbb{N}} \exists \mathrm{x} \mathrm{R}(\overline{\alpha}(\mathrm{x})) \rightarrow \exists \mathrm{y} \forall \alpha \in 2^{\mathbb{N}} \exists \mathrm{x} \leq \mathrm{y} \mathrm{R}(\overline{\alpha}(\mathrm{x})),$$

with no restrictions on the predicate R(w) except for the obvious ones on the variables. Classically (but not constructively), FT and $\mathrm{FT_d}$ are interderivable with each other and with the restriction KL of König's Lemma to the binary fan.

WKL or "Weak König's Lemma," which plays a significant role in reverse constructive mathematics, is the restriction of König's Lemma to detachable subtrees of the binary fan. Formally, WKL is obtained from KL by adding a decidability hypothesis so WKL is constructively equivalent to

$$\forall \mathrm{y} \exists \alpha \in 2^{\mathbb{N}} \forall \mathrm{x} \leq \mathrm{y} \rho(\overline{\alpha}(\mathrm{x})) = 0 \rightarrow \exists \alpha \in 2^{\mathbb{N}} \forall \mathrm{x} \rho(\overline{\alpha}(\mathrm{x})) = 0.$$

Adding a strong effective uniqueness hypothesis gives a principle WKL!:

$$\begin{aligned} &\forall \mathrm{y} \exists \alpha \in 2^{\mathbb{N}} \forall \mathrm{x} \leq \mathrm{y} \rho(\overline{\alpha}(\mathrm{x})) = 0 \\ &\quad \& \ \forall \alpha \in 2^{\mathbb{N}} \forall \beta \in 2^{\mathbb{N}} [\exists \mathrm{x} \alpha(\mathrm{x}) \neq \beta(\mathrm{x}) \rightarrow \exists \mathrm{x} \rho(\overline{\alpha}(\mathrm{x})) \neq 0 \vee \exists \mathrm{x} \rho(\overline{\beta}(\mathrm{x})) \neq 0] \\ &\qquad\qquad \rightarrow \exists \alpha \in 2^{\mathbb{N}} \forall \mathrm{x} \rho(\overline{\alpha}(\mathrm{x})) = 0. \end{aligned}$$

which is constructively equivalent to $\mathrm{FT_d}$.[2]

J. Berger's and Ishihara's round-robin proof in [2] of the equivalence of WKL! with $\mathrm{FT_d}$ implicitly used (monotone) countable choice to provide a modulus of uniform continuity for an arbitrary uniformly continuous real-valued function on Cantor space; however, their proof that WKL! entails $\mathrm{FT_d}$ was constructive in the present sense. Schwichtenberg ([9]) gave a direct proof of the equivalence in a language which anticipated conversion to a Minlog program; he then carried out the formalization in Minlog. His proof that Fan (his version of $\mathrm{FT_d}$) entails WKL! used an auxiliary proposition PFan concerning pair nodes, but did not appear to use countable choice.

[2]In this case, since both WKL! and $\mathrm{FT_d}$ are open axioms with just the variable ρ free, their constructive equivalence means that their universal closures are provably equivalent in **M**.

To verify that WKL! holds in $\mathbf{M} + \mathrm{FT_d}$, Section 2 of this note introduces and uses a separation principle SP which follows constructively from $\mathrm{FT_d}$. The remaining sections develop a third version WKL!! of Weak König's Lemma which is strictly intermediate in strength between WKL! and WKL from the present constructive standpoint.

2. Verification that $\mathrm{FT_d}$ constructively entails WKL!

The proof of $\mathrm{FT_d}$ from WKL! in [2] and [9] evidently does not need countable choice, so can be formalized in $\mathbf{M}$. The reverse entailment also holds constructively, as we now verify.

Lemma 1. $\mathrm{FT_d}$ constructively entails the separation principle

$$\mathrm{SP}: \quad \forall\alpha \in 2^{\mathbb{N}}\forall\beta \in 2^{\mathbb{N}}[\exists \mathrm{x}\, \rho(\overline{\alpha}(\mathrm{x})) \neq 0 \;\vee\; \exists \mathrm{x}\, \sigma(\overline{\beta}(\mathrm{x})) \neq 0] \\ \rightarrow \forall\alpha \in 2^{\mathbb{N}}\exists \mathrm{x}\, \rho(\overline{\alpha}(\mathrm{x})) \neq 0 \;\vee\; \forall\beta \in 2^{\mathbb{N}}\exists \mathrm{x}\, \sigma(\overline{\beta}(\mathrm{x})) \neq 0.$$

The proof requires an abbreviation for the fair merge of two sequences. For $\alpha, \beta \in 2^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}$ and for each $\mathrm{n} \in \mathbb{N}$, let $[\alpha,\beta](2\mathrm{n}) = \alpha(\mathrm{n})$ and $[\alpha,\beta](2\mathrm{n}+1) = \beta(\mathrm{n})$.

Every function τ which is defined on the set $2^* = \{0,1\}^*$ of codes of finite binary sequences and takes values in $\{0,1\}$, with the properties

(a) $\tau(\langle\,\rangle) = 0$ and
(b) $\forall \mathrm{u} \in 2^*\forall \mathrm{v} \in 2^*(\tau(\mathrm{u} * \mathrm{v}) = 0 \rightarrow \tau(\mathrm{u}) = 0)$,

is the characteristic function of a detachable subtree $\mathrm{T}_\tau = \{u \in 2^* | \tau(u) = 0\}$ of the binary tree 2^*. Conversely, every detachable subtree of 2^* is T_τ for a unique such τ. Thus (the universal closure of) $\mathrm{FT_d}$ is constructively equivalent to (the universal closure of)

$$\tau(\langle\,\rangle) = 0 \;\&\; \forall \mathrm{u} \in 2^*\forall \mathrm{v} \in 2^*(\tau(\mathrm{u} * \mathrm{v}) = 0 \rightarrow \tau(\mathrm{u}) = 0) \\ \rightarrow [\forall\alpha \in 2^{\mathbb{N}}\exists \mathrm{x}\tau(\overline{\alpha}(\mathrm{x})) \neq 0 \rightarrow \exists \mathrm{y}\forall\alpha \in 2^{\mathbb{N}}\tau(\overline{\alpha}(\mathrm{y})) \neq 0].$$

Proof of Lemma 1. Assume $\forall\alpha \in 2^{\mathbb{N}}\forall\beta \in 2^{\mathbb{N}}[\exists \mathrm{x}\, \rho(\overline{\alpha}(\mathrm{x})) \neq 0 \;\vee\; \exists \mathrm{x}\, \sigma(\overline{\beta}(\mathrm{x})) \neq 0]$. If $\rho(\langle\,\rangle) \neq 0$ or $\sigma(\langle\,\rangle) \neq 0$ there is nothing to prove. If $\rho(\langle\,\rangle) = \sigma(\langle\,\rangle) = 0$, define $\tau \in 2^{2^*}$ so that $\tau(\mathrm{u}) = 0$ if and only if either

$$\exists\alpha \in 2^{\mathbb{N}}\exists\beta \in 2^{\mathbb{N}}\exists \mathrm{k} \leq \mathrm{lh}(\mathrm{u})[\mathrm{u} = \overline{[\alpha,\beta]}(2\mathrm{k}) \\ \& \; \forall \mathrm{j} \leq \mathrm{k}\, \rho(\overline{\alpha}(\mathrm{j})) = 0 \;\&\; \forall \mathrm{j} \leq \mathrm{k}\, \sigma(\overline{\beta}(\mathrm{j})) = 0] \;\text{ or}$$

$$\exists\alpha \in 2^{\mathbb{N}}\exists\beta \in 2^{\mathbb{N}}\exists \mathrm{k} < \mathrm{lh}(\mathrm{u})[\mathrm{u} = \overline{[\alpha,\beta]}(2\mathrm{k}+1) \\ \& \; \forall \mathrm{j} \leq \mathrm{k}+1\, \rho(\overline{\alpha}(\mathrm{j})) = 0 \;\&\; \forall \mathrm{j} \leq \mathrm{k}\, \sigma(\overline{\beta}(\mathrm{j})) = 0].$$

Then τ is the characteristic function of a subtree of 2^* and $\forall\gamma \in 2^{\mathbb{N}}\exists \mathrm{y}\, \tau(\overline{\gamma}(\mathrm{y})) \neq 0$ by the hypothesis of the lemma, so by $\mathrm{FT_d}$: $\exists \mathrm{y}\forall\gamma \in 2^{\mathbb{N}}\tau(\overline{\gamma}(\mathrm{y})) \neq 0$. Let y be a witness; since $\tau(\overline{\gamma}(\mathrm{y})) \neq 0 \rightarrow \tau(\overline{\gamma}(\mathrm{y}+1)) \neq 0$, we may assume $\mathrm{y} = 2\mathrm{k}$ for some k. Then

$$(*) \quad \forall\alpha \in 2^{\mathbb{N}}\forall\beta \in 2^{\mathbb{N}}[\exists \mathrm{x} \leq \mathrm{k}\, \rho(\overline{\alpha}(\mathrm{x})) \neq 0 \;\vee\; \exists \mathrm{x} \leq \mathrm{k}\, \sigma(\overline{\beta}(\mathrm{x})) \neq 0],$$

and there are only finitely many $\mathrm{u} \in 2^*$ with $\mathrm{lh(u)} \leq \mathrm{k}$. Suppose for contradiction that $\exists\alpha \in 2^{\mathbb{N}}\forall \mathrm{x} \leq \mathrm{k}\,\rho(\overline{\alpha}(\mathrm{x})) = 0\ \&\ \exists\beta \in 2^{\mathbb{N}}\forall \mathrm{x} \leq \mathrm{k}\,\sigma(\overline{\beta}(\mathrm{x})) = 0$. Then $[\alpha,\beta] \in 2^{\mathbb{N}}\ \&$ $\forall \mathrm{x} \leq 2\mathrm{k}\,\tau(\overline{[\alpha,\beta]}(\mathrm{x})) = 0$, contradicting (*). So

$$\forall\alpha \in 2^{\mathbb{N}}\exists \mathrm{x} \leq \mathrm{k}\,\rho(\overline{\alpha}(\mathrm{x})) \neq 0\ \vee\ \forall\beta \in 2^{\mathbb{N}}\exists \mathrm{x} \leq \mathrm{k}\,\sigma(\overline{\beta}(\mathrm{x})) \neq 0,$$

and the lemma is proved.

Proposition 2. $\mathrm{FT_d}$ constructively entails WKL!

Proof. Assume ρ satisfies the hypotheses of WKL!: $\forall \mathrm{y}\exists\alpha \in 2^{\mathbb{N}}\forall \mathrm{x} \leq \mathrm{y}\rho(\overline{\alpha}(\mathrm{x})) = 0$ and $\forall\alpha \in 2^{\mathbb{N}}\forall\beta \in 2^{\mathbb{N}}[\exists \mathrm{x}\alpha(\mathrm{x}) \neq \beta(\mathrm{x}) \to \exists \mathrm{x}\rho(\overline{\alpha}(\mathrm{x})) \neq 0 \vee \exists \mathrm{x}\rho(\overline{\beta}(\mathrm{x})) \neq 0]$. For $\mathrm{w} \in 2^*$ define

$$\tau(\mathrm{w}) = 0 \leftrightarrow \forall \mathrm{u} \leq \mathrm{w}\forall \mathrm{v} \leq \mathrm{w}(\mathrm{w} = \mathrm{u} * \mathrm{v} \to \rho(\mathrm{u}) = 0),$$

otherwise $\tau(\mathrm{w}) = 1$. Then τ satisfies (a) and (b) (see the proof of Lemma 1), hence

(c) $\forall \mathrm{y}\exists\alpha \in 2^{\mathbb{N}}\,\tau(\overline{\alpha}(\mathrm{y})) = 0$ and

(d) $\forall\alpha \in 2^{\mathbb{N}}\forall\beta \in 2^{\mathbb{N}}[\exists \mathrm{x}\alpha(\mathrm{x}) \neq \beta(\mathrm{x}) \to \exists \mathrm{x}\tau(\overline{\alpha}(\mathrm{x})) \neq 0\ \vee\ \exists \mathrm{x}\tau(\overline{\beta}(\mathrm{x})) \neq 0]$.

Claim: $\forall \mathrm{n}\exists! \mathrm{u} \in 2^{\mathrm{n}}\forall \mathrm{y}\exists\alpha \in 2^{\mathbb{N}}\,\tau(\mathrm{u} * \overline{\alpha}(\mathrm{y})) = 0$.

Basis: $\mathrm{n} = 0$. Then $\forall \mathrm{y}\exists\alpha \in 2^{\mathbb{N}}\,\tau(\langle\,\rangle * \overline{\alpha}(\mathrm{y})) = 0$ by (c), and $\mathrm{u} = \langle\,\rangle$ is the unique element of 2^0.

Ind. Step: If there is a unique $\mathrm{u} \in 2^{\mathrm{n}}$ such that $\forall \mathrm{y}\exists\alpha \in 2^{\mathbb{N}}\,\tau(\mathrm{u} * \overline{\alpha}(\mathrm{y})) = 0$, then

(e) $\forall \mathrm{y}[\exists\alpha \in 2^{\mathbb{N}}\tau(\mathrm{u} * \langle 0\rangle * \overline{\alpha}(\mathrm{y})) = 0\ \vee\ \exists\beta \in 2^{\mathbb{N}}\tau(\mathrm{u} * \langle 1\rangle * \overline{\beta}(\mathrm{y})) = 0]$ by (c), and

(f) $\forall\alpha \in 2^{\mathbb{N}}\forall\beta \in 2^{\mathbb{N}}[\exists \mathrm{x}\,\tau(\mathrm{u} * \langle 0\rangle * \overline{\alpha}(\mathrm{x})) \neq 0 \vee \exists \mathrm{x}\,\tau(\mathrm{u} * \langle 1\rangle * \overline{\beta}(\mathrm{x})) \neq 0]$ from (b) and (d). Hence by SP (which follows constructively from $\mathrm{FT_d}$ by Lemma 2):

(g) $\forall\alpha \in 2^{\mathbb{N}}\exists \mathrm{x}\,\tau(\mathrm{u} * \langle 0\rangle * \overline{\alpha}(\mathrm{x})) \neq 0\ \vee\ \forall\beta \in 2^{\mathbb{N}}\exists \mathrm{x}\,\tau(\mathrm{u} * \langle 1\rangle * \overline{\beta}(\mathrm{x})) \neq 0$.

First consider the case $\forall\alpha \in 2^{\mathbb{N}}\exists \mathrm{x}\,\tau(\mathrm{u} * \langle 0\rangle * \overline{\alpha}(\mathrm{x})) \neq 0$. If $\tau(\mathrm{u} * \langle 0\rangle) \neq 0$ then $\forall \mathrm{y}\exists\beta \in 2^{\mathbb{N}}\tau(\mathrm{u} * \langle 1\rangle * \overline{\beta}(\mathrm{y})) = 0$ and $\neg\forall \mathrm{y}\exists\alpha \in 2^{\mathbb{N}}\tau(\mathrm{u} * \langle 0\rangle * \overline{\alpha}(\mathrm{y})) = 0$ by (e) and (b). If $\tau(\mathrm{u} * \langle 0\rangle) = 0$ then (a) and (b) hold with $\lambda \mathrm{v}.\tau(\mathrm{u} * \langle 0\rangle * \mathrm{v})$ in place of τ and so by $\mathrm{FT_d}$ there is a z such that $\forall\alpha \in 2^{\mathbb{N}}\exists \mathrm{x} \leq \mathrm{z}\,\tau(\mathrm{u} * \langle 0\rangle * \overline{\alpha}(\mathrm{x})) \neq 0$, whence $\forall \mathrm{y}\exists\beta \in 2^{\mathbb{N}}\tau(\mathrm{u} * \langle 1\rangle * \overline{\beta}(\mathrm{y})) = 0$ and $\neg\forall \mathrm{y}\exists\alpha \in 2^{\mathbb{N}}\tau(\mathrm{u} * \langle 0\rangle * \overline{\alpha}(\mathrm{y})) = 0$ by (e) and (b). Hence if $\forall\alpha \in 2^{\mathbb{N}}\exists \mathrm{x}\,\tau(\mathrm{u} * \langle 0\rangle * \overline{\alpha}(\mathrm{x})) \neq 0$ then $\mathrm{u} * \langle 1\rangle$ is the unique element w of $2^{\mathrm{n}+1}$ such that $\forall \mathrm{y}\exists\alpha \in 2^{\mathbb{N}}\tau(\mathrm{w} * \overline{\alpha}(\mathrm{y})) = 0$.

Similarly, if $\forall\beta \in 2^{\mathbb{N}}\exists \mathrm{x}\,\tau(\mathrm{u} * \langle 1\rangle * \overline{\beta}(\mathrm{x})) \neq 0$ then $\mathrm{u} * \langle 0\rangle$ is the unique element w of $2^{\mathrm{n}+1}$ such that $\forall \mathrm{y}\exists\alpha \in 2^{\mathbb{N}}\tau(\mathrm{w} * \overline{\alpha}(\mathrm{y})) = 0$. Thus the induction step is complete and the claim is established.

Now apply $\mathrm{AC_{00}}!$ to obtain a (unique) γ such that

$$\forall \mathrm{n}[\gamma(\mathrm{n}) \in 2^{\mathrm{n}}\ \&\ \forall \mathrm{y}\exists\alpha \in 2^{\mathbb{N}}\tau(\gamma(\mathrm{n}) * \overline{\alpha}(\mathrm{y})) = 0].$$

For each n let $\delta(\mathrm{n})$ be the last element of the sequence of length $n + 1$ coded by $\gamma(\mathrm{n} + 1)$.[3] A straightforward induction now shows that $\forall \mathrm{n}(\gamma(\mathrm{n}) = \overline{\delta}(\mathrm{n}))$, whence $\forall \mathrm{n}\,\tau(\overline{\delta}(\mathrm{n})) = 0$ and $\forall \mathrm{x}\rho(\overline{\delta}(\mathrm{x})) = 0$. So WKL! is indeed a constructive consequence of $\mathrm{FT_d}$.

[3] Using the formal versions of Kleene's coding $\langle a_0, \ldots, a_n\rangle = p_0^{a_0+1} \cdots p_n^{a_n+1}$ of finite sequences, where p_i is the ith prime number with $p_0 = 2$, and decoding in which $(v)_i$ is the exponent of p_i in the prime factorization of v, the precise definition of δ is $\delta(\mathrm{n}) = (\gamma(\mathrm{n} + 1))_{\mathrm{n}} \dot{-} 1$.

3. A stronger version of Weak König's Lemma with uniqueness

By weakening the uniqueness requirement in the hypothesis of WKL! to the (classically, but not constructively, equivalent) demand that any two witnesses to the conclusion must be equal, we obtain the schema WKL!!:

$$\begin{aligned}&\forall \mathrm{y}\exists\alpha \in 2^{\mathbb{N}}\forall \mathrm{x} \leq \mathrm{y}\rho(\overline{\alpha}(\mathrm{x})) = 0\\ &\quad \& \ \forall\alpha \in 2^{\mathbb{N}}\forall\beta \in 2^{\mathbb{N}}[\forall \mathrm{x}\rho(\overline{\alpha}(\mathrm{x})) = 0 \ \& \ \forall \mathrm{x}\rho(\overline{\beta}(\mathrm{x})) = 0 \rightarrow \forall \mathrm{x}\alpha(\mathrm{x}) = \beta(\mathrm{x})]\\ &\qquad\qquad \rightarrow \exists\alpha \in 2^{\mathbb{N}}\forall \mathrm{x}\rho(\overline{\alpha}(\mathrm{x})) = 0.\end{aligned}$$

Evidently WKL constructively entails WKL!!, and WKL!! constructively entails WKL!. Hence WKL!! constructively entails $\mathrm{FT_d}$.

To show that $\mathrm{FT_d}$ does not entail WKL!! and that WKL!! does not entail WKL, we first decompose WKL!! into a logical principle and a mathematical one, following the example of Ishihara's decomposition [4] of WKL and Berger's [1] of WKL!. Then we use this decomposition, with Kleene's function-realizability [6] and the author's ${}^{\mathrm{G}}$realizability [7], to finish the argument.

4. WKL!! is constructively equivalent to $\mathrm{MP}^{\vee}$ + ¬¬WKL

In [3] Ishihara introduced a "disjunctive version of Markov's Principle" $\mathrm{MP}^{\vee}$:

$$\neg\neg\exists \mathrm{x}(\alpha(\mathrm{x}) \neq 0 \vee \beta(\mathrm{x}) \neq 0) \rightarrow \neg\neg\exists \mathrm{x}\alpha(\mathrm{x}) \neq 0 \vee \neg\neg\exists \mathrm{x}\beta(\mathrm{x}) \neq 0,$$

and used it in a decomposition of Markov's Principle MP. Let "¬¬WKL" denote the double negation of the open axiom WKL, i.e.

$$\neg\neg[\forall \mathrm{y}\exists\alpha \in 2^{\mathbb{N}}\forall \mathrm{x} \leq \mathrm{y}\rho(\overline{\alpha}(\mathrm{x})) = 0 \rightarrow \exists\alpha \in 2^{\mathbb{N}}\forall \mathrm{x}\rho(\overline{\alpha}(\mathrm{x})) = 0],$$

or equivalently

$$\forall \mathrm{y}\exists\alpha \in 2^{\mathbb{N}}\forall \mathrm{x} \leq \mathrm{y}\rho(\overline{\alpha}(\mathrm{x})) = 0 \rightarrow \neg\neg\exists\alpha \in 2^{\mathbb{N}}\forall \mathrm{x}\rho(\overline{\alpha}(\mathrm{x})) = 0.^4$$

Theorem 3. WKL!! constructively entails $\mathrm{MP}^{\vee}$.

Proof. Assume the hypothesis of $\mathrm{MP}^{\vee}$: **(a)** $\neg\neg\exists \mathrm{x}(\alpha(\mathrm{x}) \neq 0 \vee \beta(\mathrm{x}) \neq 0)$, whence **(b)** $\neg(\forall \mathrm{x}\alpha(\mathrm{x}) = 0 \ \& \ \forall \mathrm{x}\beta(\mathrm{x}) = 0)$. Define $\rho(\mathrm{w})$ for $\mathrm{w} \in 2^*$ by setting $\rho(\mathrm{w}) = 0$ if and only if either

(i) $\mathrm{w} = \overline{\lambda \mathrm{t}.0}(\mathrm{lh}(\mathrm{w})) \ \& \ \forall \mathrm{x} < \mathrm{lh}(\mathrm{w})(\forall \mathrm{y} < \mathrm{x}\,\alpha(\mathrm{y}) = 0 \rightarrow \beta(\mathrm{x}) = 0)$, or
(ii) $\mathrm{w} = \overline{\lambda \mathrm{t}.1}(\mathrm{lh}(\mathrm{w})) \ \& \ \forall \mathrm{x} < \mathrm{lh}(\mathrm{w})(\forall \mathrm{y} \leq \mathrm{x}\,\beta(\mathrm{y}) = 0 \rightarrow \alpha(\mathrm{x}) = 0)$,

otherwise $\rho(\mathrm{w}) = 1$. Then ρ satisfies the first hypothesis of WKL!! because
(c) $\rho(\langle 0\rangle) = \rho(\langle 1\rangle) = \rho(\langle\,\rangle) = 0$,
(d) $\rho(\mathrm{u}) = 0 \rightarrow [\mathrm{u} = \overline{\lambda \mathrm{t}.0}(\mathrm{lh}(\mathrm{u})) \vee \mathrm{u} = \overline{\lambda \mathrm{t}.1}(\mathrm{lh}(\mathrm{u}))]$,
(e) $\rho(\mathrm{u} * \mathrm{v}) = 0 \rightarrow \rho(\mathrm{u}) = 0$, and
(f) $\forall \mathrm{y}\neg[\rho(\overline{\lambda \mathrm{t}.0}(\mathrm{y})) = 1 \ \& \ \rho(\overline{\lambda \mathrm{t}.1}(\mathrm{y})) = 1]$.
If γ is a witness for the conclusion of WKL!!, then
(g) $[\gamma(0) = 0 \rightarrow \forall \mathrm{x}\gamma(\mathrm{x}) = 0] \ \& \ [\gamma(0) = 1 \rightarrow \forall \mathrm{x}\gamma(\mathrm{x}) = 1]$; so either

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[4] Since double negation does not commute constructively with universal quantification, the universal closure of ¬¬WKL is weaker than the double negation of the universal closure of WKL.

(h) $\forall x\rho(\overline{\lambda t.0}(x)) = 0$ & $\neg\forall x\rho(\overline{\lambda t.1}(x)) = 0$ (so $\neg\forall x\alpha(x) = 0$, so $\neg\neg\exists x\alpha(x) \neq 0$) or

(j) $\forall x\rho(\overline{\lambda t.1}(x)) = 0$ & $\neg\forall x\rho(\overline{\lambda t.0}(x)) = 0$ (so $\neg\forall x\beta(x) = 0$, so $\neg\neg\exists x\beta(x) \neq 0$). The second hypothesis of WKL!! follows, so WKL!! guarantees the existence of a witness γ for its conclusion, hence the conclusion $\neg\neg\exists x\alpha(x) \neq 0 \vee \neg\neg\exists x\beta(x) \neq 0$ of $\mathrm{MP}^{\vee}$ holds also.

Theorem 4. WKL!! constructively entails ¬¬WKL. In fact, WKL!! $\rightarrow$ ¬¬WKL is provable in **M**.

Proof. Assume **(a)** $\forall y\exists\alpha \in 2^{\mathbb{N}}\forall x \leq y\rho(\overline{\alpha}(x)) = 0$ and

(b) $\neg\exists\alpha \in 2^{\mathbb{N}}\forall x\rho(\overline{\alpha}(x)) = 0$. Then vacuously

(c) $\forall\alpha \in 2^{\mathbb{N}}\forall\beta \in 2^{\mathbb{N}}[\forall x\rho(\overline{\alpha}(x)) = 0$ & $\forall x\rho(\overline{\beta}(x)) = 0 \rightarrow \forall x\alpha(x) = \beta(x)]$,

so ρ satisfies the hypotheses of WKL!!, so WKL!! guarantees

(d) $\exists\alpha \in 2^{\mathbb{N}}\forall x\rho(\overline{\alpha}(x)) = 0$, contradicting (b) and completing the proof.

Theorem 5. WKL!! is a constructive consequence of $\mathrm{MP}^{\vee}$ and ¬¬WKL.

Proof. Assume the hypotheses of WKL!!:

(a) $\forall y\exists\alpha \in 2^{\mathbb{N}}\forall x \leq y\rho(\overline{\alpha}(x)) = 0$ and

(b) $\forall\alpha \in 2^{\mathbb{N}}\forall\beta \in 2^{\mathbb{N}}[\forall x\rho(\overline{\alpha}(x)) = 0$ & $\forall x\rho(\overline{\beta}(x)) = 0 \rightarrow \forall x\alpha(x) = \beta(x)]$.

For each $u \in 2^*$, a primitive recursive binary functor $u * \lambda t.0$ can be defined by

$$\overline{(u * \lambda t.0)}(\mathrm{lh}(u)) = u \ \& \ \forall x \geq \mathrm{lh}(u)\ (u * \lambda t.0)(x) = 0,$$

so (a) is equivalent to **(a)**$'$: $\forall y\exists u \in 2^y\, \forall x \leq y\, \rho(\overline{(u * \lambda t.0)}(x)) = 0$. If we can prove

$$(*) \quad \forall n\exists! u \in 2^n\, \forall y\exists v \in 2^y\, \forall x < n + y\ \rho(\overline{(u * v * \lambda t.0)}(x)) = 0,$$

then $\mathrm{AC}_{00}!$ will guarantee $\exists!\alpha\forall x\rho(\overline{\alpha}(x)) = 0$.

Let $A(n, u, y)$ abbreviate $u \in 2^n$ & $\exists v \in 2^y\, \forall x < n + y\ \rho(\overline{(u * v * \lambda t.0)}(x)) = 0$. We prove $\forall n\exists! u\forall yA(n, u, y)$ by induction on n.

Basis. $\forall yA(0, \langle\,\rangle, y)$ holds by (a)$'$ since $\langle\,\rangle$ is the only sequence number of length 0. Hence $\exists! u\forall yA(0, u, y)$.

Induction Step. Assume $\exists! u\forall yA(n, u, y)$ and in particular assume $\forall yA(n, u, y)$. By ¬¬WKL:

(c) $\forall yA(n + 1, u * \langle 0\rangle, y) \rightarrow \neg\neg\exists\alpha \in 2^{\mathbb{N}}(\forall x\rho(\overline{\alpha}(x)) = 0$ & $\alpha(n) = 0)$ and

(d) $\forall yA(n + 1, u * \langle 1\rangle, y) \rightarrow \neg\neg\exists\alpha \in 2^{\mathbb{N}}(\forall x\rho(\overline{\alpha}(x)) = 0$ & $\alpha(n) = 1)$. Then

(e) $\neg[\forall yA(n + 1, u * \langle 0\rangle, y)$ & $\forall yA(n + 1, u * \langle 1\rangle, y)]$ by (b), so if γ, δ are binary functions defined by $\gamma(y) = 0 \leftrightarrow A(n, u * \langle 0\rangle, y)$ and $\delta(y) = 0 \leftrightarrow A(n, u * \langle 1\rangle, y)$ then $\neg[\forall y\, \gamma(y) = 0$ & $\forall y\, \delta(y) = 0]$, so $\neg\forall y\, \gamma(y) = 0 \vee \neg\forall y\, \delta(y) = 0$ by $\mathrm{MP}^{\vee}$, so

(f) $\neg\forall yA(n + 1, u * \langle 0\rangle, y) \vee \neg\forall yA(n + 1, u * \langle 1\rangle, y)$.

In the first case, $\forall yA(n + 1, u * \langle 1\rangle, y)$ and in the second case, $\forall yA(n + 1, u * \langle 0\rangle, y)$, by the induction hypothesis with (a)$'$. So $\exists! v\forall yA(n + 1, v, y)$, completing the proof of (*).

5. Constructively, WKL!! lies strictly between WKL! and WKL

Corollary 6. **M** + $\mathrm{FT_d}$ does not prove WKL!!. In fact, Kleene and Vesley's formal system **FIM** for intuitionistic analysis, which extends **M** + $\mathrm{FT_d}$, does not prove WKL!!.

Proof. In [7] a modified realizability notion ($^{\mathrm{G}}$realizability) was introduced and used to show the consistency of **FIM** with a weak form $\forall\alpha\neg\neg\mathrm{GR}(\alpha)$ of Church's Thesis, where $\mathrm{GR}(\alpha)$ abbreviates $\exists\mathrm{e}\forall\mathrm{x}\exists\mathrm{y}[\mathrm{T}(\mathrm{e},\mathrm{x},\mathrm{y})\ \&\ \mathrm{U}(\mathrm{y})=\alpha(\mathrm{x})]$. It follows that every theorem of **FIM**, *a fortiori* every theorem of **M** + $\mathrm{FT_d}$, is $^{\mathrm{G}}$realizable; and every constructive consequence of $^{\mathrm{G}}$realizable sentences is $^{\mathrm{G}}$realizable.

But $\mathrm{MP}^{\vee}$ is not $^{\mathrm{G}}$realizable, as we now show using notation and results from [7]. If σ $^{\mathrm{G}}$realizes $\forall\alpha\forall\beta[\neg\neg\exists\mathrm{x}(\alpha(\mathrm{x})\neq 0\vee\beta(\mathrm{x})\neq 0)\rightarrow\neg\neg\exists\mathrm{x}\alpha(\mathrm{x})\neq 0\vee\neg\neg\exists\mathrm{x}\beta(\mathrm{x})\neq 0]$ then σ is recursive and $F(\alpha,\beta)=(\{\{\{\sigma\}[\alpha]\}[\beta]\}[\Lambda\gamma.\lambda t.0](0))_0$ is a continuous function of α and β with values in $\{0,1\}$. If $F(\lambda t.0,\lambda t.0)$ depends only on $\overline{\lambda t.0}(n)$, then if $\gamma(n)=1$ and $\gamma(x)=0$ for all $x\neq n$ we must have $F(\gamma,\lambda t.0)=F(\lambda t.0,\gamma)=F(\lambda t.0,\lambda t.0)$. Since γ and $\lambda t.0$ are recursive, the hypothesis on σ requires that $F(\gamma,\lambda t.0)=0$ and $F(\lambda t.0,\gamma)=1$; contradiction. By Theorem 3 it follows that WKL!! is not a constructive consequence of $\mathrm{FT_d}$, nor of WKL! by [2].

Corollary 7. **M** + WKL!! does not prove WKL. In fact, **FIM** + WKL!! is consistent relative to **M** and does not prove WKL.

Proof. Classically, $\mathrm{MP}^{\vee}$ and $\neg\neg$WKL are Kleene function-realizable. Hence by Theorem 5, with Theorem 9.3 of [6], every theorem of **FIM** + WKL!! is Kleene function-realizable, and so **FIM** + WKL!! is consistent relative to **M** by [5]. Kleene's example ([6] Lemma 9.8) of an infinite, recursive subtree of $2^{\mathbb{N}}$ with no infinite recursive branch shows that the universal closure of WKL is not realized by any recursive function, since there is a recursive ρ for which $\forall\mathrm{y}\exists\alpha\in 2^{\mathbb{N}}\forall\mathrm{x}\leq\mathrm{y}\rho(\overline{\alpha}(\mathrm{x}))=0$ has a recursive realizer but $\exists\alpha\in 2^{\mathbb{N}}\forall\mathrm{x}\rho(\overline{\alpha}(\mathrm{x}))=0]$ has none. Hence WKL is not a constructive consequence of WKL!!.

References

1. J. Berger, *A decomposition of Brouwer's fan theorem*, Jour. of Logic and Anal. **1** (2009), 1–8.
2. J. Berger and H. Ishihara, *Brouwer's fan theorem and unique existence in constructive analysis*, Math. Log. Quart. **51** (2005), 360–364.
3. H. Ishihara, *Markov's principle, Church's thesis and Lindelöf's theorem*, Indag. Mathem., N. S. **4** (1993), 321–325.
4. ______, *Constructive reverse mathematics: Compactness properties*, From Sets and Types to Topology and Analysis: Towards Practicable Foundations for Constructive Mathematics (L. Crosilla and P. Schuster, eds.), Oxford Univ. Press, 2005, pp. 245–267.
5. S. C. Kleene, *Formalized recursive functionals and formalized realizability*, Memoirs, no. 89, Amer. Math. Soc., 1969.
6. S. C. Kleene and R. E. Vesley, *The foundations of intuitionistic mathematics, especially in relation to recursive functions*, North Holland, 1965.
7. J. R. Moschovakis, *Can there be no nonrecursive functions*, Jour. Symb. Logic **36** (1971), 309–315.
8. J. R. Moschovakis and G. Vafeiadou, *Some axioms for constructive analysis*, Preprint, available from the homepages of the authors.
9. H. Schwichtenberg, *A direct proof of the equivalence between Brouwer's fan theorem and König's lemma with a uniqueness hypothesis*, Jour. of Universal Comp. Sci. **11** (2005), 2086–2095.

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