MATH 32A.1 - FEB. 15, 2006 - J. GARNETT
MIDTERM version b

Please note: Show your work. Correct answers not accompanied by sufficent explanations will receive little or no credit. Please ask one of the proctors if you have any questions about a problem. You may use a 5 x 7 note card written on both sides, but no calculators, books, PDAs, cell phones, or other devices will be permitted. You may remove the scratch paper at the end of your exam. *If you have a question about the grading or believe that a problem has been graded incorrectly, you must bring it to the attention of your professor within 2 weeks of the exam.*

#1	#2	#3	#4	#5	Total

Section__ meets: Tuesday Thursday

TA name: ________________

SID:____________________

Name:____

Problem 1. (20 pts) Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors in 3-space.

(a) Define the projection $\vec{u} = \text{proj}_{\vec{b}}\vec{a}$, which is a vector such that $\vec{u} = c\vec{b}$ for some scalar c and $\vec{a} - \vec{u} \perp \vec{b}$. (Recall that $\perp$ means orthogonal or perpendicular.) Your answer should include a formula for the scalar c.

Solution: Take $c = \frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}$. Then the solution is $u = c\vec{b}$ because

$$(\vec{a} - \vec{u}) \cdot \vec{b} = \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{b} = 0.$$

(b) Let $\vec{u}$ be the vector in (a). Show that

$$|\vec{a} - t\vec{b}|$$

is smallest when $t\vec{b} = \vec{u}$. Hint: Write $|\vec{a} - t\vec{b}|^2$ as a dot product and use your formula for c above.

Solution: It is enough to minimize $|\vec{a} - t\vec{b}|^2 = (\vec{a} - t\vec{b}) \cdot (\vec{a} - t\vec{b})$. But by Calculus $(\vec{a} - t\vec{b}) \cdot (\vec{a} - t\vec{b}) = |\vec{a}|^2 - 2\vec{a} \cdot \vec{b}t + |\vec{b}|^2$ is smallest when $t = c$.

Problem 2. (20 pts) (a)Find a vector orthogonal (i.e. perpendicular) to the plane containing the points $(4,1,2),(2,1,1)$ and $(1,0,1)$.

Solution: The plane contains the vectors $\vec{a} =< 2,0,1 >$ and $\vec{b} =< 1,1,0 >$. Then $\vec{n} = \vec{a}\times\vec{b}$ is normal to the plane. But then

$$\vec{n} =< -1,1,2 > .$$

(b) Find the equation of the plane given in part (a).

Solution: Since the plane contains $(1,0,1)$ and has normal $\vec{n} =< -1,1,2 >$, its equation is

$$-(x-1)+y+2(z-1)=0.$$

Problem 3. (20 pts) Consider the curve

$$\vec{r}(t) = t^2\vec{i} + 2t\vec{j} + \ln t\vec{k},\ 1 \le t \le 3.$$

(a) Write down a definite integral that gives the length of this curve.

Solution: $\vec{r}'(t) =< 2t, 2, \frac{1}{t} >$ and

$$|\vec{r}'(t)| = \sqrt{4t^2 + 4 + \frac{1}{t^2}} = 2t + \frac{1}{t}.$$

The length of the curve is

$$\int_1^3 |\vec{r}'(t)|dt = \int_1^3 (2t + \frac{1}{t})dt.$$

(b) Find the length of the curve.

Solution:

$$\int_1^3 (2t + \frac{1}{t})dt = 8 + \ln 3.$$

Problem 4. (30 pts) Consider the curve

$$\vec{r}(t) =< e^t \cos t, e^t \sin t, e^t >$$

which contains the point $P = (1, 0, 1)$.

(a) For this curve and for the point P find the unit tangent vector $\vec{T}$, the normal vector $\vec{N}$, and the binormal vector $\vec{B}$.

Solution: At P we have $t = 0$. For all t,

$$\vec{r}'(t) = e^t < \cos t - \sin t, \sin t + \cos t, 1 >,$$

and $|\vec{r}'(t)| = \sqrt{3}e^t$ so that

$$\vec{T}(t) = \frac{1}{\sqrt{3}} < \cos t - \sin t, \sin t + \cos t, 1 >$$

and

$$\vec{T}(0) = \frac{1}{\sqrt{3}} < 1, 1, 1 > .$$

Then

$$\vec{T}'(t) = \frac{1}{\sqrt{3}} < -\sin t - \cos t, \cos t - \sin t, 0 >$$

and

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \frac{1}{\sqrt{2}} < -\sin t - \cos t, \cos t - \sin t, 0 > .$$

Hence

$$\vec{N}(0) = \frac{1}{\sqrt{2}} < -1, 1, 0 >$$

and

$$\vec{B}(0) = \vec{T}(0) \times \vec{N}(0) = \frac{1}{\sqrt{6}} < -1, -1, 2 > .$$

(b) Find the equation of the osculating plane to this curve at the point P.

Solution: $(-1)(x - 1) - (y - 0) + 2(z - 1) = 0$.

Problem 5. (10 pts) A moving particle $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ remains on the sphere $x^2 + y^2 + z^2 = R^2$, and has velocity $\vec{v}(t)$ for all t. Prove that $\vec{v}(t) \cdot \vec{r}(t) = 0$ for all t.

Solution: Differentiate $\vec{r}(t) \cdot \vec{r}(t) = R^2$, a constant, to get $2\vec{r}(t) \cdot \vec{r}'(t) = 0$.
Partial credit for arguing that $\vec{v}(t) = \vec{r}'(t)$ must be tangent to the sphere $x^2+y^2+z^2 = R^2$.