32A1 Practice Midterm Solutions

1. $\vec{a} = \vec{i} + \vec{j}$, $\vec{b} = 3\vec{i} + 2\vec{j} + \vec{k}$

We need $\vec{u} = c\vec{b}$, $(\vec{a} - \vec{u}) \perp \vec{b}$.

Let $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$, so $\vec{u} = c\vec{b}$ gives

$u_1 = 3c,\ u_2 = 2c,\ u_3 = c.$ $(*)$

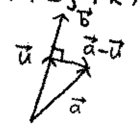
Also $(\vec{a} - \vec{u}) \perp \vec{b}$ gives $\left((1-u_1)\vec{i} + (1-u_2)\vec{j} + (-u_3)\vec{k}\right) \cdot (3\vec{i} + 2\vec{j} + \vec{k}) = 0$

$\Rightarrow 3(1-u_1) + 2(1-u_2) - u_3 = 0.$

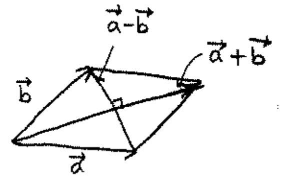
Substituting $(*)$ here, we have $3(1-3c) + 2(1-2c) - c = 0$

$\Rightarrow c = \frac{5}{14}$

$\Rightarrow \vec{u} = \frac{5}{14}(3\vec{i} + 2\vec{j} + \vec{k})$



2.



We want to show that $(\vec{a}+\vec{b}) \cdot (\vec{a}-\vec{b}) = 0$.

We are also given that length $\vec{a}$ = length $\vec{b}$

$\Leftrightarrow |\vec{a}| = |\vec{b}| \Leftrightarrow \vec{a}\cdot\vec{a} = \vec{b}\cdot\vec{b}.$ $(*)$

Now $(\vec{a}+\vec{b}) \cdot (\vec{a}-\vec{b}) = \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{a} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{b}$

$= \vec{a}\cdot\vec{a} - \vec{b}\cdot\vec{b}$ (since $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$)

$= 0$ (since $(*)$ gives $\vec{a}\cdot\vec{a} - \vec{b}\cdot\vec{b} = 0$).

3. (a) $\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\vec{k}$

$= (a_2b_3 - a_3b_2)\vec{i} - (a_1b_3 - a_3b_1)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}.$

(b) $(\vec{a} \times \vec{b}) \cdot \vec{b} = (a_2b_3 - a_3b_2)b_1 - (a_1b_3 - a_3b_1)b_2 + (a_1b_2 - a_2b_1)b_3$

$= (a_2b_1b_3 - a_2b_1b_3) + (-a_3b_1b_2 + a_3b_1b_2) + (-a_1b_2b_3 + a_1b_2b_3)$

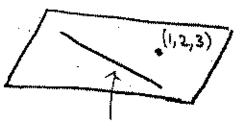
$= 0.$

4. Volume is given by $|\vec{a} \cdot \vec{b} \times \vec{c}|$ which is the absolute value of:

$$\begin{vmatrix} - \vec{a} - \\ - \vec{b} - \\ - \vec{c} - \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & -4 \end{vmatrix} = 1 \cdot \begin{vmatrix} -1 & 0 \\ 2 & -4 \end{vmatrix} - 0 \begin{vmatrix} 1 & 0 \\ 1 & -4 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix}$$

$$= 4 - 0 + 3$$
$$= 7.$$

5.



$x = 3t, y = 1+t, z = 2-t$

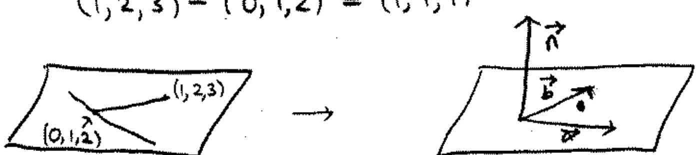
We need to find two vectors in the plane. Their cross product gives the normal vector.

The line itself gives one vector: $(3,1,-1)$ (coefficients of t).

Now pick a point on the line, say $t=0 \Rightarrow (0,1,2)$.

Form the vector between $(0,1,2)$ and $(1,2,3)$:

$$(1,2,3) - (0,1,2) = (1,1,1)$$



Now $\vec{n} = \vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 3 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 2\vec{i} - 4\vec{j} + 2\vec{k}.$

Planes are given by the normal and a point in the plane.
$(1,2,3)$ is in the plane.

$$2(x-1) - 4(y-2) + 2(z-3) = 0.$$