**Math 3A Homework #2 solutions**

**3.2:8** Let

$$f(x) = \begin{cases} \frac{x^2+x-2}{x-1} & \text{if } x \neq 1 \\ a & \text{if } x = 1 \end{cases}$$

Which value must you assign to $a$ so that $f(x)$ is continuous at $x = 1$?

**Answer:** For $f(x)$ to be continuous at $x = 1$, we require that $\lim_{x\to 1} f(x) = f(1)$. So we need

$$\begin{aligned} a &= \lim_{x\to 1} \frac{x^2 + x - 2}{x - 1} \\ &= \lim_{x\to 1} \frac{(x+2)(x-1)}{x-1} \\ &= \lim_{x\to 1} (x+2) = 3 \end{aligned}$$

So we need to set $a = 3$.

**3.2:10** Show that

$$f(x) = \begin{cases} \frac{1}{x^2-1} & \text{for } x \neq -1, 1 \\ 0 & \text{for} x = -1 \text{ or } 1 \end{cases}$$

is discontinuous at $x = -1$ and $x = 1$.

**Answer:** For $f(x)$ to be continuous at $x = \pm 1$, we need $\lim_{x\to\pm 1} f(x)$ to exist. But $\lim_{x\to\pm 1} \frac{1}{x^2-1}$ doesn't exist, since the denomenator goes to zero.

**3.2:20** Find the range of continuity of $f(x) = \cos(2x)$.

**Answer:** Since $2x$ is a polynomial, it is continuous for all $\mathbf{R}$. Since trigonometric functions are continuous wherever defined, and $\cos x$ is defined everywhere, then $\cos x$ is continuous for all $\mathbf{R}$. If two functions are continuous everywhere, then their composion is continuous everywhere. Thus $\cos(2x)$ is continuous for all $\mathbf{R}$.

**3.2:24** Find the range of continuity of $f(x) = \exp(-\sqrt{x-1})$.

**Answer:** The exponential function is continuous everywhere, but $\sqrt{x-1}$ is defined only for $x \geq 1$. So the composite function is continuous for $x \geq 1$.

**3.2:36** Find $\lim_{x\to 1} \sqrt{x^3 + 4x - 1}$.

**Answer:** The square-root function is continuous as long as its argument is non-zero, which is true when $x^3 + 4x - 1 \geq 0$. Since $1^3 + 4 \cdot 1 - 1 = 4 > 0$, $\sqrt{x^3 + 4x - 1}$ is continuous at $x = 1$. Thus $\lim_{x\to 1} \sqrt{x^3 + 4x - 1} = \sqrt{1^3 + 4 \cdot 1 - 1} = 2$.

**3.2:38** Find $\lim_{x\to 0} e^{3x+1}$.

**Answer:** The exponential function is continuous everywhere, and so is any polynomial. So $e$ raised to a polynomial is continuous everywhere. Thus $\lim_{x\to 0} e^{3x+1} = e^{3\cdot 0+1} = e^1 = e$.

**3.3:2** Find $\lim_{x\to\infty} \frac{x^2+3}{5x^2-2x+1}$.

**Answer:** Since the degree of the numerator and the degree of the denomenator are equal, the limit is just the ratio of the leading coefficients, $\frac{1}{5}$.

**3.3:8** Find $\lim_{x\to-\infty} \frac{3-x^2}{1-2x^2}$.

**Answer:** Again, the degree of numerator and denomenator are equal. So the limit is the ratio of the leading coefficients, $\frac{1}{2}$. Note that this is the same as the limit to $+\infty$.

**3.3:14** Find $\lim_{x\to\infty} \frac{e^{-x}}{1-e^{-x}}$.

**Answer:** We know that $\lim_{x\to\infty} e^{-x}$ exists, so we can use the limit laws to write

$$\begin{aligned}\lim_{x\to\infty} \frac{e^{-x}}{1-e^{-x}} &= \frac{\lim_{x\to\infty} e^{-x}}{\lim_{x\to\infty} 1 - \lim_{x\to\infty} e^{-x}} \\ &= \frac{0}{1-0} = 0\end{aligned}$$

**3.4:4c** Let $f(x) = \frac{\sin x}{x}$. Show that for $x > 0$, $-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$ holds, and use the sandwich theorem to compute $\lim_{x\to\infty} \frac{\sin x}{x}$.

**Answer:** We know that

$$-1 \leq \sin x \leq 1$$

Dividing both sides by $x$, we get

$$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

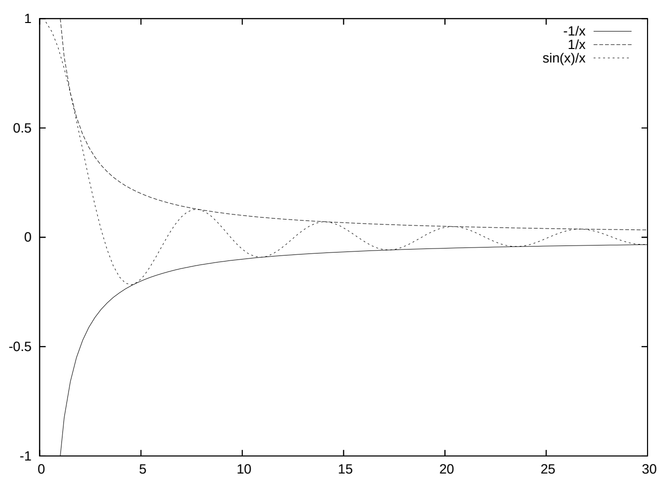
which we can do, since $x > 0$. Now we see that

$$\lim_{x\to\infty} -\frac{1}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$$

so we can apply the sandwich theorem to get

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0.$$

We can see this graphically:



**3.4:6** Find $\lim_{x\to 0}\frac{\sin(2x)}{3x}$.

**Answer:** Let $z = 2x$. Observe that $z \to 0$ as $x \to 0$. Then

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(2x)}{3x} &= \lim_{z\to 0}\frac{\sin z}{3z/2}\\ &= \frac{2}{3}\lim_{z\to 0}\frac{\sin z}{z} = \frac{2}{3}\end{aligned}$$

**3.4:12** Find $\lim_{x\to 0}\frac{\sin^2 x}{x}$.

**Answer:**

$$\begin{aligned}\lim_{x\to 0}\frac{\sin^2 x}{x} &= \lim_{x\to 0}\left(\sin x\frac{\sin x}{x}\right)\\ &= \lim_{x\to 0}\sin x\cdot\lim_{x\to 0}\frac{\sin x}{x}\\ &= 0\cdot 1 = 0\end{aligned}$$

**3.4:14** Find $\lim_{x\to 0}\frac{1-\cos^2 x}{x^2}$.

**Answer:** Using the identity $\sin^2 x + \cos^2 x = 1$, we see that

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos^2 x}{x^2} &= \lim_{x\to 0}\frac{\sin^2 x}{x^2}\\ &= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2\\ &= (1)^2 = 1\end{aligned}$$