Determinacy Proofs for Long games

Itay Neeman

Department of Mathematics University of California Los Angeles Los Angeles, CA 90095-1555 ineeman@math.ucla.edu

1.(d) Example: Σ_2^1 determinacy.

2. Games of length $\omega \cdot \omega$ with Σ_2^1 payoff.

3. Continuously coded games with Σ_2^1 payoff.

Recall: A is the set of all reals which satisfy a given Σ_2^1 statement ϕ . $\dot{A} \in M$ names the set of reals of $M^{\operatorname{col}(\omega,\delta)}$ which satisfy ϕ in $M^{\operatorname{col}(\omega,\delta)}$.

G is the game in which I and II play $x = \langle x_0, x_1, \ldots \rangle \in \mathbb{R}$ and in addition play moves in the auxiliary game $\mathcal{A}[x]$.

Using $\sigma_{\rm piv}$ to ascribe auxiliary moves for II we showed that

Case 1. If I wins G in M, then (in V) I has a winning strategy in $G_{\omega}(A)$.

Let \dot{B} in M name the set of reals which do **not** satisfy ϕ in $M^{\operatorname{col}(\omega,\delta)}$.

Define $x \mapsto \mathcal{B}[x]$ and $x \mapsto \mathcal{B}^*[x]$ as before, but changing \dot{A} to \dot{B} and interchanging I and II.

We have $\tau_{gen}[x,g]$ and $\tau_{piv}[x,g]$ as before, but with the roles of I and II switched.

Let H be the following game, defined and played inside M:

I and II alternate playing natural numbers, producing $x = \langle x_0, x_1, \ldots \rangle \in \mathbb{R}$. In addition they play moves $b_{0-I}, b_{0-II}, \ldots$ in $\mathcal{B}[x]$.

This time I is the closed player; she wins if she can last all ω moves. Otherwise II wins.

Case 2: II wins H. Then an argument similar to that of Case 1 shows that (in V) II has a strategy to get into $B = \mathbb{R} - A$. In other words, II wins $G_{\omega}(A)$ in V. \Box (Case 2.)

We showed:

- If I wins G in M, then (in V) I wins $G_{\omega}(A)$.
- If II wins H in M, then (in V) II wins $G_{\omega}(A)$.

It is now enough to check that one of these cases must occur.

Suppose not. I.e., assume that, in M, II wins G and I wins H. Fix strategies $\Sigma^{II} \in M$ and $\Sigma^{I} \in M$ witnessing this. We wish to derive a contradiction.

Recall the progress of the games G and H:

G:	Ι	x_0	a_{0-I}		a_{1-I}	• • •
	II		a_{0-II}	x_1	$a_{1-\mathrm{II}}$	
H:	Ι	x_0	b_{0-I}		b_{1-I}	• • •
	II		$b_{0-\mathrm{II}}$	x_1	$b_{1-\mathrm{II}}$	

Working in M[g], construct $x = \langle x_0, x_1, \ldots \rangle$, $\vec{a} = \langle a_{0-I}, a_{0-II}, \ldots \rangle$, and $\vec{b} = \langle b_{0-I}, b_{0-II}, \ldots \rangle$ as follows:

- Σ^{II} (playing for II in G) produces x_n for odd n, and $a_{n-\text{II}}$ for all n.
- $\sigma_{gen}[x,g]$ produces a_{n-I} for all n.
- Σ^{I} (playing for I in H) produces x_{n} for even n and b_{n-I} for all n.
- $\tau_{\text{gen}}[x,g]$ produces $b_{n-\text{II}}$ for all n.

We get $x \notin \dot{A}[g]$ by Lemma 1. Similarly we get $x \notin \dot{B}[g]$ through our use of τ_{gen} .

But \dot{A} and \dot{B} name complementary sets. Since $x \in M[g]$ this is a contradiction.

To sum: Defined in M the game

where I, II collaborate to produce $x \in \mathbb{R}$, and in addition play auxiliary moves: I trying to witness $x \in \dot{A}[h]$ for some h, II trying to witness the opposite. G is a closed game.

If in M I wins G, showed (using σ_{piv}) that in V I wins to get into some $j_b(\dot{A})[h]$, and hence by absoluteness into A.

Defined in M the game

This time II is trying to witness $x \in \dot{B}[h]$ for some h, and I is trying to witness the opposite.

If in M II wins H, showed (using τ_{piv}) that in V II wins to get into some $j_b(\dot{B})[h]$, and hence by absoluteness into $B = \mathbb{R} - A$.

Finally, if both cases fail, we worked in M[g] (using σ_{gen} and τ_{gen}) to construct $x \in M[g]$ which belongs to neither $\dot{A}[g]$ nor $\dot{B}[g]$, a contradiction.

Fix $C \subset \mathbb{R}^{\omega}$ a Σ_2^1 set, say the set of all sequences $\langle y_0, y_1, \ldots \rangle \in \mathbb{R}^{\omega}$ which satisfy the Σ_2^1 statement ϕ .

We wish to prove that $G_{\omega \cdot \omega}(C)$ is determined.

Fix M and an increasing sequence $\delta_1, \delta_2, \ldots, \delta_\omega$ so that

- *M* is a class model.
- Each δ_{ξ} is a Woodin cardinal in M.
- In V there is g which is $col(\omega, \delta_{\omega})$ -gen/M.
- *M* is iterable.

The existence of such M is our large cardinal assumption (needed to prove determinacy). We use δ_{∞} and g_{∞} to refer to δ_{ω} and g. Let $\dot{A}_{\infty} \in M$ name the set of elements of $\mathbb{R}^{\omega} \cap M[g_{\infty}]$ which satisfy ϕ in $M[g_{\infty}]$.

For $\langle y_n \mid n < \omega \rangle \in \mathbb{R}^{\omega}$ we have the associated game $\mathcal{A}_{\infty}[y_n \mid n < \omega]$. (Formally we should think of $\langle y_n \mid n < \omega \rangle$ as coded by some real x.)

The association is continuous, and we may talk about $\mathcal{A}_{\infty}[y_0, \ldots, y_{k-1}]$, a game of k+1 rounds.

We use a_{0-I}^{∞} , a_{0-II}^{∞} , a_{1-I}^{∞} , etc. to refer to moves in \mathcal{A}_{∞} .

We use a_n^{∞} to denote $\langle a_{n-\mathrm{I}}^{\infty}, a_{n-\mathrm{II}}^{\infty} \rangle$ and refer to runs of \mathcal{A}_{∞} as \vec{a}^{∞} .

(Recall that moves in $\mathcal{A}_{\infty}[y_n \mid n < \omega]$ are arranged so that I tries to witness $\langle y_n \mid n < \omega \rangle \in \dot{A}_{\infty}[h]$ for some h, and II tries to witness the opposite.)

A k-sequences is a sequence

$$\langle y_0, \ldots, y_{k-1}, a_0^\infty, \ldots, a_{k-1}^\infty, \gamma \rangle$$

so that

- Each y_i is a real;
- $a_0^{\infty}, \ldots, a_{k-1}^{\infty}$ is a position in the auxiliary game $\mathcal{A}_{\infty}[y_0, \ldots, y_{k-1}]$; and
- γ is an ordinal.

We use S to denote k-sequences.

A valid extension for a k-sequence is a triplet $y_k, a_k^{\infty}, \gamma^*$ so that

- y_k is a real;
- $a_k^{\infty} = \langle a_{k-\mathrm{I}}^{\infty}, a_{k-\mathrm{II}}^{\infty} \rangle$ where $a_{k-\mathrm{I}}^{\infty}$ and $a_{k-\mathrm{II}}^{\infty}$ are legal moves for I and II respectively in the game $\mathcal{A}_{\infty}[y_0, \ldots, y_{k-1}]$,* following the position $a_0^{\infty}, \ldots, a_{k-1}^{\infty}$; and
- γ^* is an ordinal **smaller** than γ .

$$\langle y_0,\ldots,y_{k-1},y_k,a_0^\infty,\ldots,a_{k-1}^\infty,a_k^\infty,\gamma^*\rangle.$$

*Observe that knowledge of y_k is not needed to determine the rules for round k of this game.

For expository simplicity, fix for each n some g_n which is $col(\omega, \delta_n)$ -generic/M. Do this so that the sequence $\langle g_n | n < \omega \rangle$ belongs to $M[g_\infty]$ and each g_n belongs to $M[g_{n+1}]$.

Below we define sets in $M[g_n]$ where strictly speaking we should be defining names in $M^{\operatorname{col}(\omega,\delta_n)}$.

We work to define sets A_k in $M[g_k]$ $(k \ge 1)$. A_k will be a set of k-sequences.

Given $a_0^{\infty}, \ldots, a_{k-1}^{\infty}$, γ we let $A_k[a_0^{\infty}, \ldots, a_{k-1}^{\infty}, \gamma]$ be the set of tuples $\langle y_0, \ldots, y_{k-1} \rangle$ so that

$$\langle y_0, \ldots, y_{k-1}, a_0^\infty, \ldots, a_{k-1}^\infty, \gamma \rangle \in A_k.$$

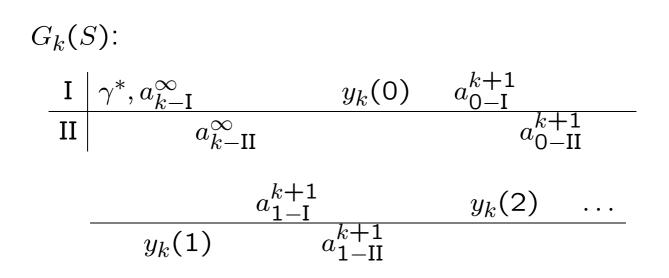
 $A_k[a_0^{\infty}, \ldots, a_{k-1}^{\infty}, \gamma]$ then is a subset of \mathbb{R}^k in $M[g_k]$. Really we are defining names, not sets. So we have a name $\dot{A}_k[a_0^{\infty}, \ldots, a_{k-1}^{\infty}, \gamma]$. Let $\mathcal{A}_k[y_0, \ldots, y_{k-1}, a_0^{\infty}, \ldots, a_{k-1}^{\infty}, \gamma]$ be auxiliary game associated to $\langle y_0, \ldots, y_{k-1} \rangle$ and the name $\dot{A}_k[a_0^{\infty}, \ldots, a_{k-1}^{\infty}, \gamma]$.

We use $\mathcal{A}_k[S]$ to denote this game, and use $a_{0-\mathrm{I}}^k$, $a_{0-\mathrm{II}}^k$ etc. to denote moves in the game.

(Recall that these moves are such that I tries to witness that S belongs to $\dot{A}_k[h]$ for some h. II tries to witness the opposite.)

Given $S = \langle y_0, \ldots, y_{k-1}, a_0^{\infty}, \ldots, a_{k-1}^{\infty}, \gamma \rangle$, a *k*-sequence, define a game $G_k(S)$ in which:

I and II play a valid extension $\gamma^*, a_k^{\infty}, y_k$. In addition I tries to witness that the extended sequence, S—, $y_k, a_k^{\infty}, \gamma^*$, belongs to A_{k+1} . II tries to witness the opposite.



- I and II play
 - γ^* ,

•
$$a_k^{\infty} = \langle a_{k-\mathrm{I}}^{\infty}, a_{k-\mathrm{II}}^{\infty} \rangle$$
, and

•
$$y_k = \langle y_k(0), y_k(1), \ldots \rangle$$

which form a valid extension of S. (In particular γ^* is **smaller** than γ .)

In addition they play auxiliary moves in the game $\mathcal{A}_{k+1}[S-, y_k, a_k^{\infty}, \gamma^*]$.

II is the closed player; she wins if she can last ω moves. Otherwise I wins.

Define the sets A_k by:

 $S \in A_k$ iff I has a winning strategy in $G_k(S)$

(for a k-sequence $S \in M[g_k]$).

If S belongs to A_k , we expect to be able to extend to $S^* = S - , y_k, a_k^{\infty}, \gamma^*$ which belongs to a "shift" of A_{k+1} .

Our definition of A_k depends on some knowledge of A_{k+1} . (We need knowledge of G_k , which involves the auxiliary game \mathcal{A}_{k+1} .)

The definition is by induction, not on k, but on $\gamma.$

Figuring out the rules of $G_k(S)$, where $S = \langle *, \ldots, *, \gamma \rangle$, requires knowledge of the sets $A_{k+1}[a_0^{\infty}, \ldots, a_k^{\infty}, \gamma^*]$, but only for $\gamma^* < \gamma$.

Determining whether S belongs to A_k thus requires knowledge of A_{k+1} , but only for k + 1-sequences ending with $\gamma^* < \gamma$.

A 0-sequence is simply an ordinal γ . We have for each γ the game $G_0(\gamma)$. This game belongs to M.

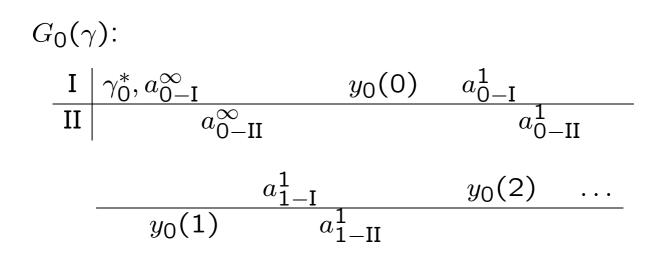
Case 1: There exists some γ so that (in M) I has a winning strategy in $G_0(\gamma)$.

We will show that (in V) I has a winning strategy in $G_{\omega \cdot \omega}(C)$.

Fix $\Sigma_0 \in M$, a winning strategy for I (the open player) in $G_0(\gamma)$.

Fix an imaginary opponent, playing for II in $G_{\omega \cdot \omega}(C)$.

We will use Σ_0 , the strategies $\sigma_{\text{piv}-1}, \sigma_{\text{piv}-2}, \ldots$, the strategy $\sigma_{\text{piv}-\infty}$, and an iteration strategy for M, to play against the imaginary opponent.

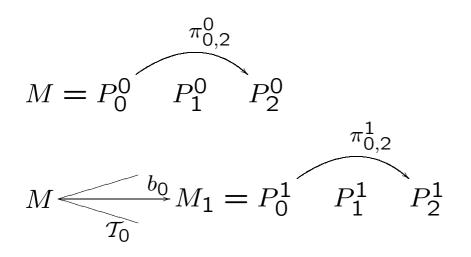


Our opponent, Σ_0 , $\sigma_{piv-\infty}$ (for the first round), and σ_{piv-1} (for the remaining rounds) cover all moves in the game.

We obtain an iteration tree \mathcal{U}^0 of length 3, played by $\sigma_{\text{piv}-\infty}$, with final model P_2^0 , embedding $\pi_{0,2}^0: M \to P_2^0$, and moves $\bar{\gamma}_0^*, \bar{a}_0^\infty$ in P_2^0 .

We obtain $y_0 \in \mathbb{R}$, and an iteration tree \mathcal{T}_0 (played by σ_{piv-1}) with illfounded even model.

The iteration strategy picks an odd branch, b_0 say. Let M_1 be the direct limit along b_0 and let $j_{0,1}$ be the direct limit embedding.



Let $\mathcal{U}^1 = j_{0,1}(\mathcal{U}^0)$, and similarly with P_2^1 , $\pi_{0,2}^1$. Let $\gamma_0^* = j_{0,1}(\bar{\gamma}_0^*)$ and similarly a_0^∞ .

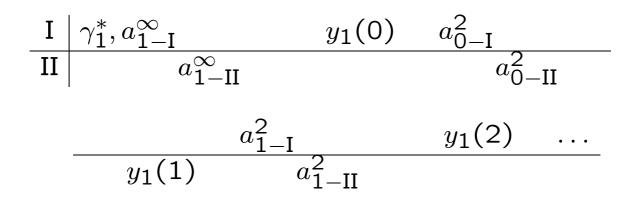
Our use of $\sigma_{\text{piv}-1}$ guarantees that there exists some h_1 so that

1. h_1 is col(ω, δ_1^s)-generic/ M_1 , and

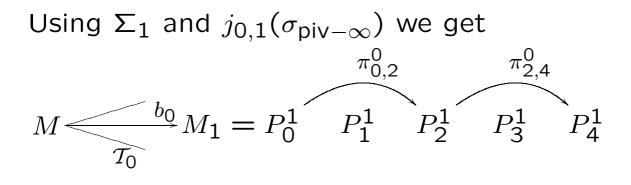
2.
$$\langle y_0, a_0^{\infty}, \gamma_0^* \rangle \in \dot{A}_1^{s}[h_1].$$

$$(*^{s} \text{ denotes } j_{0,1}(\pi^{0}_{0,2}(*)).)$$

Note that by 2, player I (the open player) has a winning strategy in $G_1^s(y_0, a_0^\infty, \gamma_0^*)$. Fix $\Sigma_1 \in M_1[h_1]$, a strategy for I witnessing this.



Note, Σ_1 belongs to $M_1[h_1]$, a **small** generic extension of M. (Small with respect to δ_2 and δ_{∞} .) This allows us to shift Σ_1 along the even branch of trees given by $\sigma_{piv-\infty}$ and σ_{piv-2} .



Then using $j_{0,1}(\sigma_{\rm piv-2})$ and **shifts** of Σ_1 get $\pi_{0,4}^2$

$$M \underbrace{\overset{b_0}{\longleftarrow}}_{T_0} M_1 \underbrace{\overset{b_1}{\longleftarrow}}_{T_1} M_2 = P_0^2 \qquad P_4^2$$

(where $P_0^2 = j_{1,2}(P_0^1)$, etc.).

We get γ_1^* , a_1^∞ , and y_1 . Our use of $\sigma_{\rm piv-2}$ guarantees that there exists h_2 so that

- 1. h_2 is col(ω, δ_2^{ss})-generic/ M_2 , and
- 2. $\langle y_0, y_1, a_0^{\infty-s}, a_1^{\infty}, \gamma_1^* \rangle \in \dot{A}_2^{ss}[h_2].$

(A second ^s stands for application of $j_{1,2} \circ \pi^1_{2,4}$.)

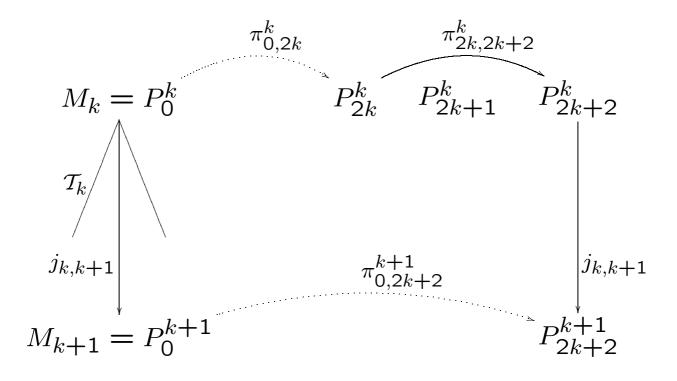
By 2, player I (the open player) wins

$$G_2^{ss}(y_0, y_1, a_0^{\infty-s}, a_1^{\infty}, \gamma_1^*).$$

This game belongs to $M[h_2]$. Fix $\Sigma_2 \in M[h_2]$, a strategy witnessing that I wins.

Continue as before.

In general we have:

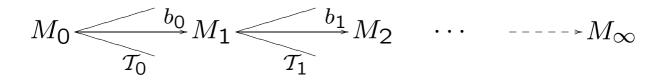


In P_{2k}^k we have the k-sequence $S_k = \langle y_0, \dots, y_{k-1}, a_0^{\infty - \dots s}, \dots, a_{k-1}^{\infty}^{-\dots -}, \gamma_{k-1}^* \rangle.$

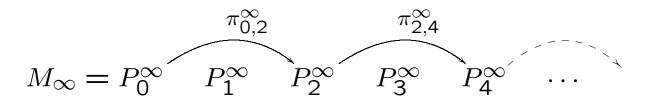
 S_{k+1} (in P_{2k+2}^{k+1}) is obtained as a valid extension of $j_{k,k+1}(\pi_{2k,2k+2}^k(S_k))$. In particular:

(†) γ_k^* is smaller than $j_{k,k+1}(\pi_{2k,2k+2}^k(\gamma_{k-1}^*))$.

We end with a sequence of reals $\langle y_n \mid n < \omega \rangle$, a sequence of iteration trees



and an iteration tree \mathcal{U}_{∞} on M_{∞} as follows:



By (†) the even branch of \mathcal{U}_{∞} is illfounded.

The iteration strategy for M produces an odd branch c of \mathcal{U}_{∞} . Let M_c be the direct limit, and let $\pi_c: M_{\infty} \to M_c$ be the direct limit embedding. Note M_c , played by an iteration strategy, is wellfounded. Now \mathcal{U}_{∞} is part of a play according to $j_{0,\infty}(\sigma_{\mathsf{piv}-\infty})[y_n \mid n < \omega].$

Our use of $j_{0,\infty}(\sigma_{\mathsf{piv}-\infty})[y_n \mid n < \omega]$ guarantees that there exists some h_∞ so that

- 1. h_{∞} is col($\omega, \pi_c(j_{0,\infty}(\delta_{\infty})))$ -generic/ M_c , and
- 2. $\langle y_n \mid n < \omega \rangle \in \pi_c(j_{0,\infty}(\dot{A}_\infty))[h_\infty].$

From 2 we see that $\langle y_n | n < \omega \rangle$ satisfies the Σ_2^1 statement ϕ , inside $M_c[h_\infty]$.

By absoluteness ϕ is satisfied in V.

So $\langle y_n \mid n < \omega \rangle \in C$ and I won, as required.

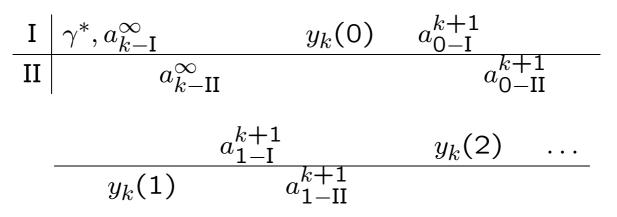
 \Box (Case 1.)

Assuming there is some γ so that (in M) I wins $G_0(\gamma)$, we showed that (in V) I wins $G_{\omega \cdot \omega}(C)$.

Fix $\gamma_{\rm L} < \gamma_{\rm H}$ indiscernibles for M, above δ_{∞} .

Suppose $S = \langle y_0, \dots, y_{k-1}, a_0^{\infty}, \dots, a_{k-1}^{\infty}, \gamma_{\mathsf{L}} \rangle$ is a *k*-sequence in $M[g_k]$ and does not belong A_k .

So II wins $G_k(S)$. By indiscernibility II also wins $G_k(S_{\mathsf{H}})$ where $S_{\mathsf{H}} = \langle *, \dots, \gamma_{\mathsf{H}} \rangle$. Fix a winning strategy $\Sigma_{\mathrm{II}-k}$.



Play $\gamma^* = \gamma_L$. Use Σ_{II-k} , $\sigma_{gen-\infty}$, $\sigma_{gen-(k+1)}$ to obtain a_k^{∞} , \vec{a}^{k+1} , and (half of) y_k in $M[g_{k+1}]$.

Our use of $\sigma_{\text{gen}-(k+1)}$ guarantees that $S' = \langle S_{\text{H}}, y_k, a_k^{\infty}, \gamma_{\text{L}} \rangle \notin \dot{A}_{k+1}[g_{k+1}].$

If S' belongs to $M[g_{k+1}]$ this means that II wins $G_{k+1}(S')$.

Continue this way. Our use of $\sigma_{\text{gen}-\infty}$ guarantees that $\langle y_n \mid n < \omega \rangle$ does not belong to $\dot{A}[g_{\infty}]$.

If there is γ so that I wins the closed game $G_0(\gamma)$, then I has a winning strategy in $G_{\omega \cdot \omega}(C)$.

Mirroring this with sets B_k and games H_k we get:

If there is γ so that II wins the closed game $H_0(\gamma)$, then II has a winning strategy in $G_{\omega \cdot \omega}(C)$.

Finally, if II wins $G_0(\gamma_{\rm L})$ and I wins $H_0(\gamma_{\rm L})$, we can work in $M[g_n]$, $n < \omega$, and produce $\langle y_n \mid n < \omega \rangle \in M[g_\infty]^*$ which belongs to neither $\dot{A}[g_\infty]$ nor $\dot{B}[g_\infty]$, a contradiction.

It follows that $G_{\omega \cdot \omega}(C)$ is determined.

*Note $\langle g_n \mid n < \omega \rangle \in M[g_{\infty}].$