DETERMINACY FOR GAMES ENDING AT THE FIRST ADMISSIBLE RELATIVE TO THE PLAY

ITAY NEEMAN

Abstract. Let $o(\kappa)$ denote the Mitchell order of κ. We show how to reduce long games which run to the first ordinal admissible in the play, to iteration games on models with a cardinal κ so that (1) κ is a limit of Woodin cardinals; and (2) $o(\kappa) = \kappa^{++}$. We use the reduction to derive several optimal determinacy results on games which run to the first admissible in the play.

Given a set $C \subset \mathbb{R}^{<\omega_1}$ consider the following game, denoted $G_{\mathrm{adm}}(C)$: In mega-round ξ players I and II alternate natural numbers as in Diagram 1, producing together a real $y_\xi = \langle y_\xi(n) \mid n < \omega \rangle$. They continue this way until reaching the first ordinal α so that $\mathrm{L}_\alpha[y_\xi \mid \xi < \alpha]$ is *admissible*. At that point the game ends. Player I wins if $\langle y_\xi \mid \xi < \alpha \rangle \in C$, and otherwise player II wins.

$$\begin{array}{c|cccccccccc}
\mathrm{I} & y_0(0) & & y_0(2) & & \ldots\ldots & y_\xi(0) & & y_\xi(2) & & \ldots\ldots \\
\hline
\mathrm{II} & & y_0(1) & & \ldots & & & y_\xi(1) & & \ldots &
\end{array}$$

Diagram 1. The game $G_{\mathrm{adm}}(C)$.

We develop in this paper methods for proving the determinacy of $G_{\mathrm{adm}}(C)$ for appropriately definable C, from optimal large cardinal assumptions. The methods involve combining the techniques of Neeman [5] for proofs of determinacy of long games, with the rank games of Neeman [6]. The large cardinals used depend on the complexity of the payoff set C, but in all cases they involve a cardinal κ which is a limit of Woodin cardinals, and has Mitchell order κ^{++}. (The Woodin cardinals are used in the proof in connection with the methods of Neeman [5], while the measures are used in connection with the rank games.)

Section 2 presents the rank games that we need in this paper, and the relevant results. Most interesting among them is Lemma 2.10 (whose proof makes use of the measures involved in the large cardinal assumptions). Section 3 explains how we reduce G_{adm} to an *iteration* game on models with sufficient large cardinals. The actual proof of the reduction is spread over Sections 4, 5, 6, and 7. Finally Section 8 uses the reduction to prove several optimal results concerning the determinacy of G_{adm}.

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§1. Codes. We begin by describing how to code positions in G_{adm} by reals.

By a **position** in G_{adm} we mean a sequence of reals $\langle x_\xi \mid \xi < \alpha\rangle$ so that for each $\beta < \alpha$, $\mathcal{J}_\beta[x_\xi \mid \xi < \beta]$ is not admissible. $\mathcal{J}_\alpha[x_\xi \mid \xi < \alpha]$ itself may or may not be admissible. If it is, then we say that the position is **terminal**.

Remark 1.1. By $\mathcal{J}_\beta[x_\xi \mid \xi < \beta]$ we mean the model $\mathcal{J}^A_\beta$ where A is the predicate defined by $\langle \omega\cdot\xi + n, m\rangle \in A$ iff $x_\xi(n) = m$. For sufficiently closed ordinals β this is the same as $\mathrm{L}_\beta[x_\xi \mid \xi < \beta]$. The $\mathcal{J}$ hierarchy is more convenient for our purposes, mostly for reasons of indexing: the ordinal height of $\mathcal{J}^A_\beta$ is $\omega\cdot\beta$, and these are precisely the ordinals we need to convert $\langle x_\xi \mid \xi < \beta\rangle$ into the predicate A.

By a **precode** we mean a pair $x = \langle w, f\rangle$ where w is a linear order on ω and $f\colon \omega \to \mathbb{R}$ is a partial function whose domain equals the domain of w.

Remark 1.2. Fix some recursive injection $\varphi\colon \mathrm{LO}\times\mathbb{R}^\omega \to \mathbb{R}$ with the property that $(\forall n)\ \varphi(w,f)\restriction n$ depends only on $w\restriction n\times n$ and $f(0)\restriction n, \dots, f(n-1)\restriction n$. Abusing notation we often confuse $\varphi(x)$ with x—for example we say that x and x^* agree to n just in case that $\varphi(x)\restriction n = \varphi(x^*)\restriction n$—and in general view precodes as *reals*.

We say that a precode x is **wellfounded** just in case that w is wellfounded. We use o.t.(w) to denote the order type of w in this case. If w is wellfounded then we think of $\langle w, f\rangle$ as coding a countable sequence $\vec{x} = \langle x_\xi \mid \xi < \alpha\rangle$ of reals, where $\alpha = \text{o.t.}(w)$, and for each $\xi < \alpha$, x_ξ is equal to $f(n)$ for the unique n whose order type in w is ξ. We refer to α as the **length** of x, denoted $\mathrm{lh}(x)$.

A wellfounded precode $x = \langle w, f\rangle$ for the sequence $\langle x_\xi \mid \xi < \alpha\rangle$ induces naturally an enumeration of all the elements of $\mathcal{J}_\alpha[\vec{x}]$. Let us make this precise.

Fix an injection $(*,\dots,*)$ of $\omega^{<\omega}$ into $\omega - \{0\}$, and an enumeration $\{F_e\}_{e<\omega}$ of all finite compositions of rudimentary functions (these are the functions that generate $\mathcal{J}^A_{\gamma+1}$ from $\mathcal{J}^A_\gamma \cup \{J^A_\gamma\}$ for each γ and all A, see Jensen [1] or Zeman [9, p.2]). We define a 1-1 coding function $c\colon \mathcal{J}_\alpha[\vec{x}] \to \omega$, working by induction on $\gamma < \alpha$.

Let $a \in \mathcal{J}_{\gamma+1}[\vec{x}] - \mathcal{J}_\gamma[\vec{x}]$. Then there exists an $e < \omega$, an $i < \omega$, and $m_0,\dots,m_i < \omega$ so that

$$a = F_e(c^{-1}(m_0),\dots,c^{-1}(m_i), J_\gamma[\vec{x}])$$

with $c^{-1}(m_0),\dots,c^{-1}(m_i) \in \mathcal{J}_\gamma[\vec{x}]$. Define $c(a)$ to be the smallest $n < \omega$ which equals $(k, n, e, m_0, \dots, m_i)$ where $k < \omega$ is the unique number whose order type in w is γ, and $e, m_0, \dots, m_i$ satisfy the equation above.

Working by induction on $\gamma \le \alpha$ one can verify that $a \mapsto c(a)$ is a 1-1 function from $\mathcal{J}_\gamma[\vec{x}\restriction\gamma]$ into ω. There is a dependence on $\vec{x}$ in the definition, since the functions F_e are allowed to refer to A. But for $a \in \mathcal{J}_\gamma[\vec{x}]$ only $\vec{x}\restriction\gamma$ affects the definition of $c(a)$.

Let us next begin to consider admissibility. Let $\{\phi_e\}_{e<\omega}$ enumerate all the formulae of set theory. If the structure $\mathcal{J}_\alpha[\vec{x}]$ and all its initial segments are not admissible then there exists an $e < \omega$, an $i < \omega$, and $m_0,\dots,m_i < \omega$ so that:

- For every $n < \omega$ there exists an ordinal $\beta < \alpha$ so that

  $$\mathcal{J}_{\beta+1}[\vec{x}\restriction\beta] \models \phi_e[n, c^{-1}(m_0),\dots,c^{-1}(m_i), \vec{x}\restriction\beta].$$

  We let β_n denote the least such ordinal; and

- The ordinals β_n are increasing and cofinal in α.

The least $l < \omega$ which equals $(e, m_0, \dots, m_i)$ for a tuple satisfying the above is said to **meter** the precode x.

Remark 1.3. It is easy to see that the same number l can*not* meter both $x = \langle w, f\rangle$ and a precode x' for a strict initial segment of $\vec{x}$.

For $k \in \mathrm{dom}(w)$ we use $x{\restriction}k$ to denote the pre-code $x' = \langle w', f'\rangle$, where w' is the restriction of w to numbers w–below k, and f' is the restriction of f to the domain of w'. This is a precode for the sequence $\langle x_\xi \mid \xi < \gamma\rangle$ where γ is the order type of k in w.

Definition 1.4. A **code** is a wellfounded precode $x = \langle w, f\rangle$, for a sequence $\langle x_\xi \mid \xi < \alpha\rangle$ say, which satisfies the following additional requirements:

- $\langle x_\xi \mid \xi < \alpha\rangle$ is a position in G_{adm}, meaning that for each $\beta < \alpha$, $\mathcal{J}_\beta[\vec{x}]$ is not admissible; and
- For each $k \in \mathrm{dom}(w)$, k meters $x{\restriction}k$.

A **terminal** code is a code for a terminal position in G_{adm}.

Remark 1.5. For each $\beta < \alpha$ let k_β be the unique number whose order type in w is β. Notice that the last condition in Definition 1.4 determines k_β uniquely, from knowledge of $\langle x_\xi \mid \xi < \beta\rangle$ and $\langle k_\xi \mid \xi < \beta\rangle$: k_β must be the unique number which meters the code obtained from these sequences.

If x is not terminal then there exists a unique $l < \omega$ which meters x, and by Remark 1.3 this l is *not* in the domain of w. Thus given some $y \in \mathbb{R}$ we can define a code $x^* = \langle w^*, f^*\rangle$ for the position $\langle x_\xi \mid \xi < \alpha\rangle^\frown\langle y\rangle$ as follows: let w^* include w and the additional relations "l greater than j" for all $j \in \mathrm{dom}(w)$; and let f^* include f and the additional assignment $l \mapsto y$.

Definition 1.6. We use $x—, y$ to denote the code x^* defined above.

Remark 1.7. Let $\langle x_\xi \mid \xi < \alpha\rangle$ be a position in G_{adm}. Then there exists exactly one x which codes this position. This can be proved by induction on α. Uniqueness follows from the previous remark, and existence is proved using the definition of $x—, y$ above.

Using the last remark we can freely switch between codes and positions, and use terminology defined for one to apply for the other. For example we say that k meters a position $\langle x_\xi \mid \xi < \alpha\rangle$ iff it meters the unique code for this position.

Let x be a code, say for the sequence $\langle x_\xi \mid \xi < \alpha\rangle$. We say that $l = (e, m_0, \dots, m_i)$ is **expected** at x, or that l is the expectation at $\vec{x} = \langle x_\xi \mid \xi < \alpha\rangle$, just in case that the following conditions hold:

1. There exists $n < \omega$ such that:
   (a) for each $\bar{n} \leq n$ there is $\beta \leq \alpha$ so that
   $$\mathcal{J}_{\beta+1}[\vec{x}{\restriction}\beta] \models \phi_e[\bar{n}, c^{-1}(m_0), \dots, c^{-1}(m_i), \vec{x}{\restriction}\beta].$$
   Let $\beta_{\bar{n}}$ denote the least such ordinal.
   (b) The ordinals $\beta_{\bar{n}}$ ($\bar{n} \leq n$) are increasing, and $\beta_n = \alpha$.
2. l is the least number for which condition (1) holds.

Remark 1.8. Recall that the injection $(*,\ldots,*)\colon \omega^{<\omega} \to \omega$ that we are using for the coding here maps $\omega^{<\omega}$ into $\omega - \{0\}$. It follows from this that the expectation at x is never equal to 0.

Claim 1.9. *Let x be a code of limit length, for the position $\vec{x} = \langle x_\xi \mid \xi < \alpha\rangle$ say. Then $\mathcal{J}_\alpha[\vec{x}]$ is admissible if and only if there is* no *fixed number l which is expected at $\vec{x}\restriction\beta$ for cofinally many $\beta < \alpha$. Moreover, if $\mathcal{J}_\alpha[\vec{x}]$ is not admissible, then l meters x if and only if (a) l is expected at $\vec{x}\restriction\beta$ for cofinally many $\beta < \alpha$; and (b) l is the least number for which (a) holds.* ⊣

The last claim begins to deal with the relationship between a code of limit length and its initial segments. Let us now see how codes converge to a terminal code.

Definition 1.10. Let x and x^* be codes, for the positions $\vec{x} = \langle x_\xi \mid \xi < \alpha\rangle$ and $\vec{x}^* = \langle x^*_\xi \mid \xi < \alpha^*\rangle$ say. x^* is an n–**extension** of x just in case that:

1. The sequence $\vec{x}^*$ extends the sequence $\vec{x}$;
2. For each $\eta \in [\alpha, \alpha^*)$, the unique number which meters $\vec{x}^*\restriction\eta$ is greater than or equal to n; and
3. For each $\eta \in [\alpha, \alpha^*)$, the expectation at $\vec{x}\restriction\eta$ is greater than or equal to n.

Claim 1.11. *If x^* is an n–extension of x then x^* and x agree to n.* ⊣

Claim 1.12. *Let x^* be an n–extension of x, and suppose x^* is not terminal. Let m meter x^* and let l be the expectation at x^*. Let y be a real. Suppose that $m \geq n$ and $l \geq n$. Then x^*—$, y$ is an n–extension of both x and x^*.* ⊣

Claim 1.13. *Let α be a limit ordinal. Let $\langle y_\xi \mid \xi < \alpha\rangle$ be a position. For each $\beta \leq \alpha$ let x_β code $\langle y_\xi \mid \xi < \beta\rangle$. Then $x_\beta \to x_\alpha$ as $\beta \to \alpha$.*

Proof. Let $j < \omega$ be given. We wish to show that a tail-end of $\langle x_\eta \mid \eta < \alpha\rangle$ agrees with x_α to j.

Since each number $m < j$ can only meter x_η for one $\eta < \alpha$, and since α is a limit, we can find $\beta < \alpha$ so that for each $\eta \in [\beta, \alpha)$, the number which meters x_η is $\geq j$. It follows from this that x_η and x_α agree to j, for every $\eta \in [\beta, \alpha)$. ⊣

Claim 1.14. *Let α be a limit ordinal. Let $\langle y_\xi \mid \xi < \alpha\rangle$ be a position. Let $\langle \alpha_n \mid n < \omega\rangle$ be an increasing sequence of ordinals, cofinal in α. Let x_n code $\langle y_\xi \mid \xi < \alpha_n\rangle$. Suppose that for each $n > 0$, x_n is an n–extension of x_{n-1}. Then $\langle y_\xi \mid \xi < \alpha\rangle$ is terminal.*

Proof. Since x_n is an n–extension of x_{n-1}, no number below n is expected at $\langle y_\xi \mid \xi < \beta\rangle$ for $\beta \in [\alpha_{n-1}, \alpha_n)$. It follows that there is no number l which is expected at $\langle y_\xi \mid \xi < \beta\rangle$ for cofinally many $\beta < \alpha$. By Claim 1.9, $\langle y_\xi \mid \xi < \alpha\rangle$ is terminal. ⊣

§2. Rank games.

We describe here an adaptation of the rank games of [6]. We will need these adapted games and the related results later on.

Let κ be a cardinal. We work in V, though we will often refer to the forcing $\mathrm{col}(\omega, {<}\kappa)$ over V. We assume throughout this section that:

(∗) for every $Z \subset \mathrm{V}_{\kappa+1}$ there exists a measure μ on κ so that $Z \in \mathrm{Ult}(\mathrm{V}, \mu)$.

Definition 2.1. By a **location** we mean a pair $\langle p, \dot{x}\rangle$ so that p is a condition in $\mathrm{col}(\omega, {<}\kappa)$, and $\dot{x} \in \mathrm{V}_\kappa$ is a name which is forced by p to be a code. A location $\langle p^*, \dot{x}^*\rangle$ is an n–**extension** of the location $\langle p, \dot{x}\rangle$ just in case that $p^* \leq p$ and p^* forces $\dot{x}^*$ to be an n–extension of $\dot{x}$ (see Definition 1.10).

Let $f\colon \mathrm{V}_\kappa \to \mathrm{V}_\kappa - \{\emptyset\}$ be the bijection defined by $f(x) = x$ if $x \notin \omega$, and $f(x) = 1 + x$ if $x \in \omega$.

Definition 2.2. Let $W \subset \mathrm{V}_{\kappa+1}$. Let $\langle p, \dot{x}\rangle$ be a location. Define the $(p, \dot{x})$–**section** of W to be the set of all $U \subset \mathrm{V}_\kappa$ so that $\{\langle p, \dot{x}\rangle\} \times (f''U \cup \{\emptyset\})$ belongs to W.

The definition lets us join V_κ many subsets of $\mathrm{V}_{\kappa+1}$ into one, in the following precise sense: Suppose that for each location $\langle p, \dot{x}\rangle$ we have associated a set $W_{p,\dot{x}} \subset \mathrm{V}_{\kappa+1}$. Then there is a set $W \subset \mathrm{V}_{\kappa+1}$ so that for each location $\langle p, \dot{x}\rangle$, $W_{p,\dot{x}}$ is precisely equal to the $(p, \dot{x})$–section of W. (This can be seen easily by setting $W = \{\ \{\langle p, \dot{x}\rangle\} \times (f''U \cup \{\emptyset\}) \mid U \in W_{p,\dot{x}}\}$.)

Remark 2.3. The use of the bijection $f\colon \mathrm{V}_\kappa \to \mathrm{V}_\kappa - \{\emptyset\}$ in Definition 2.2 is needed for arguments of the kind given in the previous paragraph. It is important there that $\emptyset$ is not put into W, and for this reason we replace U by $f''U \cup \{\emptyset\}$. But from now on we ignore this, abusing notation and writing $\{\langle p, \dot{x}\rangle\} \times U$ where strictly speaking we should write $\{\langle p, \dot{x}\rangle\} \times (f''U \cup \{\emptyset\})$.

Definition 2.4. Let $W \subset \mathrm{V}_{\kappa+1}$, let $\langle p, \dot{x}\rangle$ be a location. The **basic rank game** associated to κ, W, and $\langle p, \dot{x}\rangle$ is played according to the following rules:

- Player I starts the game by playing U in the $(p, \dot{x})$–section of W.
- Player II ends the game by playing some pair $\langle \bar{\kappa}, \bar{W}\rangle$ so that: $\bar{\kappa}$ is an inaccessible cardinal smaller than κ, and larger than the Von Neumann ranks of p and $\dot{x}$; $\bar{W} \subset \mathrm{V}_{\bar{\kappa}+1}$; and $\langle \bar{\kappa}, \bar{W}\rangle \in U$.

$$\begin{array}{c|cc} \mathrm{I} & U & \\ \hline \mathrm{II} & & \langle \bar{\kappa}, \bar{W}\rangle \end{array}$$

Diagram 2. The basic rank game.

Notice that the basic rank game ends very much as it started—with $\bar{\kappa}$, $\bar{W}$, and $\langle p, \dot{x}\rangle$ in the same relationship as κ, W, and $\langle p, \dot{x}\rangle$—but with $\bar{\kappa}$ smaller than κ. $\bar{\kappa}$ and $\bar{W}$ are chosen by player II, but I can regulate the choice, through the requirement $\langle \bar{\kappa}, \bar{W}\rangle \in U$ in the second item. U in turn is regulated by the initial set W, and the location $\langle p, \dot{x}\rangle$.

Definition 2.5. Let $W \subset \mathrm{V}_{\kappa+1}$. In the **inverted rank game** associated to κ and W players I and II collaborate to create, among other things, a sequence of locations $\langle p_n, \dot{x}_n\rangle$ $(n < \omega)$ so that each $n > 0$, $\langle p_n, \dot{x}_n\rangle$ is an n–extension of $\langle p_{n-1}, \dot{x}_{n-1}\rangle$. We set $W_0 = W$, let $p_0 = \emptyset$ and let $\dot{x}_0$ name the code for the empty position. The game proceeds according to Diagram 3 and the following format, beginning with round 1.

- At the start of round n we have a location $\langle p_{n-1}, \dot{x}_{n-1}\rangle$, and a set $W_{n-1} \subset \mathrm{V}_{\kappa+1}$.
- Player II plays a location $\langle p_n, \dot{x}_n\rangle$ and a set $U_n \subset \mathrm{V}_\kappa$ so that: p_n forces a value for $\dot{x}_n{\restriction}n$, $\langle p_n, \dot{x}_n\rangle$ is an $(n-1)$–extension of $\langle p_{n-1}, \dot{x}_{n-1}\rangle$, and U_n belongs to the $(p_n, \dot{x}_n)$–section of W_{n-1}.
- Player I plays μ_n and W_n so that μ_n is a measure on κ, $W_n \subset \mathrm{V}_{\kappa+1}$, and $\langle \kappa, W_n\rangle$ belongs to $i_{\mu_n}(U_n)$. This ends round n.

I	....		μ_n, W_n	......
II	....	$p_n, \dot{x}_n, U_n$		......

Diagram 3. Round n in the inverted rank game.

Notice the reversal of roles in the inverted rank game, compared to the basic rank game. In the inverted game it is player II who plays the set U_n, while player I must come up with W_n. Player I is better off here than player II was in the case of the basic rank game, as she is *not* asked to play $\bar{\kappa} < \kappa$. Instead of having to move below κ, she gets to send κ up through the ultrapower embedding by a measure of her choice. (This is illustrated in Diagram 4.)

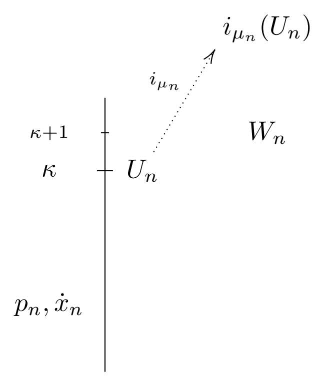

Diagram 4. Pushing U_n up.

Definition 2.6. We use T_W to denote the tree of the inverted rank game associated to κ and W.

Definition 2.7. Let r be a position in T_W. Let $p_n, \dot{x}_n, U_n$ be the final move for II in r. Define $\pi_W(r)$ to be the value of $\dot{x}_n{\restriction}n$ forced by p_n. π_W is then a map from T_W into $\omega^{<\omega}$. It gives rise to a Lipschitz continuous embedding $\pi_W\colon [T_W] \to \mathbb{R}$, sending $\vec{r} \in [T_W]$ to $\bigcup_{n<\omega} \pi_W(\vec{r}{\restriction}n)$. (In viewing $\pi_W(\vec{r})$ as a real we are identifying each precode x with its corresponding real, see Remark 1.2.)

A position r in the inverted rank game is called **whole** if it ends with a complete round (as opposed to just moves for II in that round). If r is a whole position, covering rounds 1 through $n-1$ say, then we refer to the triple $\langle p_{n-1}, \dot{x}_{n-1}, W_{n-1}\rangle$ (see the first item in Definition 2.5) as the **ending** of r.

LEMMA 2.8. *Let Ψ be a strategy for* I *in (at least the first n rounds of) the inverted rank game associated to κ and W. Let r be a whole position in the game, covering rounds* 0 *through $n-1$ say, played according to Ψ. Let $\langle p_{n-1}, \dot{x}_{n-1}, W_{n-1}\rangle$ be the ending of r.*

Let $\langle p_n, \dot{x}_n\rangle$ be an $(n-1)$–extension of $\langle p_{n-1}, \dot{x}_{n-1}\rangle$, with p_n forcing a value for $\dot{x}_n{\restriction}n$. Let G denote the basic rank game associated to κ, W_{n-1}, and $\langle p_n, \dot{x}_n\rangle$. Let Σ be a strategy for I *in G.*

Then there are U_n, μ_n, and W_n so that:

1. *$r^\frown\langle p_n, \dot{x}_n, U_n, \mu_n, W_n\rangle$ is a legal extension of r by one round in the inverted rank game, played according to Ψ; and*
2. *$i_\mu(U_n)$ and $\langle \kappa, W_n\rangle$ form a play of $i_\mu(G)$, according to $i_\mu(\Sigma)$.*

PROOF. Let U_n be Σ's move in G. Then let μ_n and W_n be the moves played by Ψ following $r^\frown\langle p_n, \dot{x}_n, U_n\rangle$. ⊣

Lemma 2.8 illustrates the reversal of roles discussed above. It shows how strategies for player I in the basic and inverted rank games can be combined, to produce complete rounds in both: I's move in the basic rank game G doubles as a move for II in the inverted rank game, and I's move in the inverted rank game doubles as a move for II in the basic rank game (shifted by i_μ).

We shall need also a dual to Lemma 2.8, dealing with the case that Ψ and Σ are strategies for player II. This dual is much more intricate than Lemma 2.8 (whose proof was nothing more than a simple combination of Ψ and Σ). Let us start with some definitions.

Work with some fixed W, and a fixed strategy Ψ for II in (at least the first n rounds of) the inverted rank game associated to κ and W.

A position r in the inverted rank game is **medial** if the last round in r covers only the moves for player II. Let r be a medial position, covering rounds 1 through the first half of round n say, and ending with the move $\langle p_n, \dot{x}_n, U_n\rangle$ for II. We refer to $\langle p_n, \dot{x}_n\rangle$ as the **end location** of r. We say that $\langle p_{n+1}, \dot{x}_{n+1}, U_{n+1}\rangle$ is **reachable** from r just in case that there exists a legal moves $\langle \mu_n, W_n\rangle$ for I following r, so that Ψ's reply to $r^\frown\langle \mu_n, W_n\rangle$ consists of p_{n+1}, $\dot{x}_{n+1}$ and U_{n+1}. Otherwise we say that $\langle p_{n+1}, \dot{x}_{n+1}, U_{n+1}\rangle$ is **unreachable** from r.

Let $\mathrm{unrch}(r)$ be the set $\{\ \{\langle p, \dot{x}\rangle\} \times U \mid \langle p, \dot{x}\rangle$ is a location, $U \subset \mathrm{V}_\kappa$, and $\langle p, \dot{x}, U\rangle$ is unreachable from $r\}$.

CLAIM 2.9. *Let r be a medial position played according to Ψ. Then there does not exist a measure μ so that $\langle \mu, \mathrm{unrch}(r)\rangle$ is a legal move for* I *in the inverted rank game following r.*

PROOF. Suppose for contradiction that $\langle \mu, \mathrm{unrch}(r)\rangle$ is legal, and play this move for I following r. Let $\langle p, \dot{x}, U\rangle$ be Ψ's reply. The rules of the inverted rank game (specifically the second item in Definition 2.5) are such that U belongs to the $(p, \dot{x})$–section of $\mathrm{unrch}(r)$. In other words $\{\langle p, \dot{x}\rangle\} \times U$ belongs to $\mathrm{unrch}(r)$, so $\langle p, \dot{x}, U\rangle$ is not reachable from r. But this is a contradiction, since there *is* a move for I following r that causes Ψ to reply with p, $\dot{x}$, and U, namely the move $\langle \mu, \mathrm{unrch}(r)\rangle$. ⊣

The next lemma is a parallel of Lemma 2.8 for the case of strategies for player II.

Lemma 2.10. *Let Ψ be a strategy for player* II *in (at least the first n rounds of) the inverted rank game associated to κ and W. Let r be a medial position in the game, played according to Ψ.*

Let $\langle p, \dot{x}\rangle$ be a location. Let G denote the basic rank game associated to κ, $\mathrm{unrch}(r)$*, and $\langle p, \dot{x}\rangle$. Let Σ be a strategy for* II *in G.*

Then there are μ, U, and r^* so that:

1. *r^* is a medial extension of r by one round in the inverted rank game, played according to Ψ; and*
2. *U and $\langle \kappa, \mathrm{unrch}(r^*)\rangle$ form a play of $i_\mu(G)$, according to $i_\mu(\Sigma)$.*

Proof. Let Y be the set:

$$\{\ \langle \tau, B\rangle \ \mid \ \tau < \kappa,\ B \subset \mathrm{V}_{\tau+1}, \text{ and there does } \textit{not} \text{ exist any } U \text{ which is legal for player I in } G \text{ and so that } \Sigma\text{'s reply to } U \text{ is } \langle \tau, B\rangle.\ \}$$

Y is thus the set of pairs which are *not* played by Σ.

Claim 2.11. *Y does not belong to the $(p, \dot{x})$–section of* $\mathrm{unrch}(r)$*. (In other words $\{\langle p, \dot{x}\rangle\} \times Y$ does not belong to* $\mathrm{unrch}(r)$*.)*

Proof. Suppose that it does. By the rules of the basic rank game, Y is then a legal move for player I in G. Play this move, and let $\langle \tau, B\rangle$ be the reply given by Σ. The rules of the basic rank game, specifically the rules in the second item of Definition 2.4, demand that $\langle \tau, B\rangle \in Y$. But this contradicts the definition of Y, since there *is* a legal move U for I which causes Σ to reply with $\langle \tau, B\rangle$, namely $U = Y$. ⊣

Corollary 2.12. *There is a medial extension r^* of r by one round in the inverted rank game associated to κ and W so that r^* is according to Ψ, and so that* II*'s (namely Ψ's) final move in r^* is $\langle p, \dot{x}, Y\rangle$.*

Proof. This is immediate from the last claim and the definition of $\mathrm{unrch}(r)$: From the fact that $\{\langle p, \dot{x}\rangle\} \times Y$ does not belong to $\mathrm{unrch}(r)$ it follows that $\langle p, \dot{x}, Y\rangle$ *is* reachable from r, so there is a move for player I following r that makes Ψ reply with $\langle p, \dot{x}, Y\rangle$. ⊣

Fix r^* as in the last corollary. Note that $\mathrm{unrch}(r^*)$ is a subset of $\mathrm{V}_{\kappa+1}$. Using the large cardinal assumption $(*)$ from the beginning of this section we can therefore fix a measure μ on κ so that $\mathrm{unrch}(r^*)$ belongs to $\mathrm{Ult}(\mathrm{V}, \mu)$.

Claim 2.13. *$\langle \kappa, \mathrm{unrch}(r^*)\rangle$ does not belong to $i_\mu(Y)$.*

Proof. Suppose that it does. The rules of the inverted rank game are such that $\langle \mu, \mathrm{unrch}(r^*)\rangle$ is then a legal move for player I following r^*. But this is in contradiction to Claim 2.9. ⊣

Let W^* denote $\mathrm{unrch}(r^*)$. By the choice of μ, we know that $\langle \kappa, W^*\rangle$ belongs to $\mathrm{Ult}(\mathrm{V}, \mu)$. But by the last claim, $\langle \kappa, W^*\rangle$ does not belong to $i_\mu(Y)$. From the definition of Y it follows that there *is* a legal move U for player I in $i_\mu(G)$ which causes $i_\mu(\Sigma)$ to reply with $\langle \kappa, W^*\rangle$. U and $\langle \kappa, W^*\rangle = \langle \kappa, \mathrm{unrch}(r^*)\rangle$ then form a play of $i_\mu(G)$ according to $i_\mu(\Sigma)$. This completes the proof of Lemma 2.10. ⊣

Remark 2.14. It's worthwhile pointing out the use of the fact that W^* belongs to $\mathrm{Ult}(\mathrm{V}, \mu)$ in the previous paragraph (and through it the use of the large cardinal assumption $(*)$). Without this fact we wouldn't be able to derive anything from the knowledge that $\langle \kappa, W^* \rangle \notin i_\mu(Y)$: $\langle \kappa, W^* \rangle$ could fail to belong to $i_\mu(Y)$ simply because it fails to belong to $\mathrm{Ult}(\mathrm{V}, \mu)$.

Remark 2.15. Notice that we had some freedom in the choice of μ during the proof of Lemma 2.10: we just needed the measure μ to be strong enough that $\mathrm{unrch}(r^*)$ belongs to $\mathrm{Ult}(M, \mu)$. The statement of Lemma 2.10 can thus be strengthened to say that: there exists a set $A \subset \mathrm{V}_{\kappa+1}$, so that for any μ strong enough that $A \in \mathrm{Ult}(M, \mu)$, there are U and r^* so that conditions (1) and (2) in Lemma 2.10 hold.

We will use Lemmas 2.8 and 2.10 later on. Let us end this section with some words on pairs reachable from the *empty* position.

Let X be a collection of subsets of $\mathrm{V}_{\kappa+1}$. Suppose that for each $W \in X$ we have a strategy Ψ_W for player II in the inverted rank game associated to κ and W. We say that $\langle p, \dot{x}, U \rangle$ is **reachable** from the empty position (relative to the collection of strategies $\{\Psi_W\}_{W \in X}$) if there exists some $W \in X$ so that Ψ_W's first move is precisely $\langle p, \dot{x}, U \rangle$. Otherwise we say that $\langle p, \dot{x}, U \rangle$ is **unreachable**.

We use $\mathrm{unrch}(\emptyset)$ to denote the set $\{ \{\langle p, \dot{x} \rangle\} \times U \mid \langle p, \dot{x} \rangle$ is a location, $U \subset \mathrm{V}_\kappa$, and $\langle p, \dot{x}, U \rangle$ is unreachable from the empty position$\}$.

The following Claim is a parallel of Claim 2.9 to the case of the empty position.

CLAIM 2.16. $\mathrm{unrch}(\emptyset)$ *does* not *belong to* X.

PROOF. Suppose for contradiction that $\mathrm{unrch}(\emptyset)$ belongs to X. Then we have a strategy $\Psi_{\mathrm{unrch}(\emptyset)}$. Let $\langle p, \dot{x}, U \rangle$ be the first move played by this strategy. The rules of the inverted rank game are such that U belongs to the $(p, \dot{x})$–section of $\mathrm{unrch}(\emptyset)$. In other words $\{\langle p, \dot{x} \rangle\} \times U$ belongs to $\mathrm{unrch}(\emptyset)$. But this contradicts the definition of $\mathrm{unrch}(\emptyset)$, since there *is* $W \in X$ so that Ψ_W's first move is $\langle p, \dot{x}, U \rangle$, namely $W = \mathrm{unrch}(\emptyset)$. ⊣

§3. Outline. Let M be a model of a sufficiently large fragment of ZFC^-, and let κ be a cardinal of M so that:

(A) κ is a limit of Woodin cardinals in M;
(B) For every $Z \subset M \| \kappa + 1$ in M there exists a measure $\mu \in M$ on κ, so that $Z \in \mathrm{Ult}(M, \mu)$; and
(C) $M \| \kappa + 1$ is countable in V.

Condition (B) is simply the relativization of the large cardinal assumption $(*)$ of the previous section, to M.

In the next few sections we will define game trees D and E in M, together with embeddings $\rho \colon D \to \omega^{<\omega}$ and $\chi \colon E \to \omega^{<\omega}$ (giving rise to Lipschitz embeddings, which we also denote ρ and χ, from $[D]$ and $[E]$ respectively, into $\mathbb{R}$). The results below are stated with reference to these trees and embeddings.

Let Ψ be a strategy for player I on D. Suppose that Ψ is **close** to M, in the sense that for every $n < \omega$, the restriction of Ψ to positions of length n belongs to M. Note that in this case there is a natural way to apply elementary embeddings on M to Ψ. More precisely, let Ψ_n be the restriction of Ψ to positions of length n. Given an embedding $j\colon M \to M^*$, we use $j(\Psi)$ to mean $\bigcup_{n<\omega} j(\Psi_n)$.

Define $\widehat{G}_{\mathrm{adm}}(\Psi, M, \kappa)$, or $\widehat{G}_{\mathrm{adm}}$ for short, to be played according to the rules below, starting with mega-round 1.

At the start of mega-round α for α a successor ordinal, we have a sequence of reals $\langle y_\xi \mid \xi < \alpha - 1\rangle$, a model M^α which is an iterate of $M = M^1$, and an iteration embedding $j^{1,\alpha}\colon M \to M^\alpha$. The mega-round itself is played as follows:

1. To start the two players alternate playing natural numbers in the usual fashion, producing together the real $y_{\alpha-1}$.
2. Player I then plays a measure μ^α in M^α and a length ω iteration tree $\mathcal{T}^\alpha$ on $\mathrm{Ult}(M^\alpha, \mu^\alpha)$.
3. Player II ends the mega-round playing a cofinal branch b^α through $\mathcal{T}^\alpha$. We let $M^{\alpha+1}$ be the direct limit along b^α, let $j^{\alpha,\alpha+1}\colon M^\alpha \to M^{\alpha+1}$ equal $j_{b^\alpha} \circ i_{\mu^\alpha}$ where i_{μ^α} is the ultrapower embedding by μ^α and j_{b^α} is the direct limit embedding along b^α, define the remaining embeddings $j^{\zeta,\alpha+1}$ (for $\zeta < \alpha$) by composition, and pass to the next mega-round.

At the start of mega-round α for α a limit ordinal we have a sequence of reals $\langle y_\xi \mid \xi < \alpha\rangle$ and an iteration $\langle M^\xi, j^{\zeta,\xi} \mid \zeta \le \xi < \alpha\rangle$ of $M^1 = M$. Let M^α be the direct limit of this iteration, and let $j_{\zeta,\alpha}$ be the direct limit maps. Mega-round α is played according to rules (2) and (3) above, giving rise to $M^{\alpha+1}$ and $j^{1,\alpha+1}$.

$\widehat{G}_{\mathrm{adm}}$ continues until reaching the first α so that $\mathcal{J}_\alpha[y_\xi \mid \xi < \alpha]$ is admissible. At that point the game ends. We let x code the sequence $\langle y_\xi \mid \xi < \alpha\rangle$, let M^* denote M^α, and let $j\colon M \to M^*$ denote $j^{1,\alpha}$.

Player I wins the run of $\widehat{G}_{\mathrm{adm}}$ described above just in case that there exists an infinite branch $\vec{d}$ through $j(D)$, so that:

(P1) $\vec{d}$ is according to $j(\Psi)$; and
(P2) $j(\rho)(\vec{d})$ is equal to x.

In other words, player I wins just in case that there is a play according to $j(\Psi)$, which projects to x.

Let Ω be a strategy for II on E. Suppose that Ω is close to M. Define $\widehat{H}_{\mathrm{adm}}(\Omega, M, \kappa)$, or $\widehat{H}_{\mathrm{adm}}$ for short, to be played according to the rules of $\widehat{G}_{\mathrm{adm}}$ above, except that in $\widehat{H}_{\mathrm{adm}}$ player II plays the measures μ^α and the iteration trees $\mathcal{T}^\alpha$, and player I plays the branches b^α. A run of $\widehat{H}_{\mathrm{adm}}$ is won by player II just in case that there exists an infinite branch $\vec{e}$ through $j(E)$, so that:

(Q1) $\vec{e}$ is according to $j(\Omega)$; and
(Q2) $j(\chi)(\vec{e})$ is equal to x.

Notice that the payoff here is for player II. It mirrors precisely the payoff for I in $\widehat{G}_{\mathrm{adm}}$.

In the next few sections we will prove the following three lemmas on the games $\widehat{G}_{\mathrm{adm}}$ and $\widehat{H}_{\mathrm{adm}}$:

Lemma 3.1. *Let Ψ be a strategy for player* I *in D, and suppose that Ψ is close to M. Then player* I *has a winning strategy in the game $\widehat{G}_{\mathrm{adm}}(\Psi, M, \kappa)$.*

Lemma 3.2. *Let Ω be a strategy for player* II *in E, and suppose that Ω is close to M. Then player* II *has a winning strategy in the game $\widehat{H}_{\mathrm{adm}}(\Omega, M, \kappa)$.*

Lemma 3.3. *Let Ψ be a strategy for player* II *in D, and let Ω be a strategy for player* I *in E. (Note the reversal compared to the previous lemmas: here we take a strategy for* II *in D, and for* I *in E.) Suppose that Ψ and Ω are close to M. Then there is (in* V*) an infinite branch $\vec{d}$ through D, and an infinite branch $\vec{e}$ through E so that:*

- *$\vec{d}$ is according to Ψ;*
- *$\vec{e}$ is according to Ω; and*
- *$\rho(\vec{d}) = \chi(\vec{e})$.*

In the next section we define D and ρ. In Section 5 we prove Lemma 3.1. In Section 6 we briefly say how to mirror the work on D, ρ, and Lemma 3.1, so as to obtain E, χ, and Lemma 3.2. In Section 7 we prove Lemma 3.3.

Finally in Section 8 we use the three lemmas to prove determinacy for long games ending at the first admissible. The reader may skip directly to this section, to see right away how the lemmas are used.

§4. Basic definitions.

We work with M and κ which satisfy the assumptions of the previous section. Our goal is to define the tree D and the map $\rho\colon D \to \omega^{<\omega}$.

Let $\bar{\kappa} < \kappa$. Let δ be the first Woodin cardinal of M above $\bar{\kappa}$. For expository simplicity fix a $\mathrm{col}(\omega, \delta)$–generic g over M. Let $\dot{A}$ be a $\mathrm{col}(\omega, \delta)$–name for a set of codes in $M[g]$. Let $\dot{A}'$ be the canonical name for $(M\|\delta)^\omega \times \dot{A}[g]$. Let $X = M\|\bar{\kappa}$. By the **auxiliary games map for $\bar{\kappa}$ and $\dot{A}$** we mean the auxiliary games map associated to $\dot{A}'$, δ, and X, as defined in Neeman [5, §1A]. We refer the reader to Chapter 1 of [5] for complete details. We will use the results of this chapter in the proofs here.

Remark 4.1. In the definition above we are as usual viewing codes as reals, through the identification $x \mapsto \varphi(x)$, see Remark 1.2. Both φ and the auxiliary games map are continuous in such a way that the rules for the first n rounds in $\mathcal{A}[x]$ depend only on $x{\restriction}n$ (or equivalently on $\varphi(x){\restriction}n$).

Definition 4.2. Let x be a code in a small generic extension of M. A **rank progression** wrt x consists of finite sequences $\langle \kappa_i, W_i, p_i, \dot{x}_i \mid i \leq n\rangle$ and $\langle \dot{A}_i, P_i \mid 0 < i \leq n\rangle$ satisfying the following conditions:

1. $\kappa_0 = \kappa$ and $\kappa_{i+1} < \kappa_i$ for each $i < n$;
2. $W_i \subset \mathrm{V}_{\kappa_i+1}$;
3. Each $\langle p_i, \dot{x}_i\rangle$ is a location belonging to V_{κ_i}. $\langle p_0, \dot{x}_0\rangle$ is the empty location (meaning that $p_0 = \emptyset$ and $\dot{x}_0$ names the code for the empty position), and for each $i > 0$, $\langle p_i, \dot{x}_i\rangle$ is an $(i-1)$–extension of $\langle p_{i-1}, \dot{x}_{i-1}\rangle$.
4. For each i, p_i forces the value of $\dot{x}_i{\restriction}i$ to be precisely $x{\restriction}i$;
5. $\dot{A}_i$ is a $\mathrm{col}(\omega, \delta_i)$–name for a set of codes, where δ_i is the first Woodin cardinal of M above κ_i; and
6. P_i is a finite position (consisting of complete rounds) in $\mathcal{A}_i[x]$, where $\mathcal{A}_i$ is the auxiliary games map for κ_i and $\dot{A}_i$.

For short we use $\{\kappa_i, W_i, p_i, \dot{x}_i, \dot{A}_i, P_i\}_{i\leq n}$ to denote the rank progression. We refer to n as the length of this progression.

By a **rank progression** we mean a sequence $\{\kappa_i, W_i, p_i, \dot{x}_i, \dot{A}_i, P_i\}_{i\leq n}$ which satisfies the conditions above for some x. A rank progression **fits** x if it is a rank progression wrt x. Notice that only $x\restriction \max\{n, \mathrm{lh}(P_1), \ldots, \mathrm{lh}(P_n)\}$ is relevant to Definition 4.2. If x and x^* agree to $\max\{n, \mathrm{lh}(P_1), \ldots, \mathrm{lh}(P_n)\}$, then x fits $\{\kappa_i, W_i, p_i, \dot{x}_i, \dot{A}_i, P_i\}_{i\leq n}$ iff x^* does.

Definition 4.3. Let $\vec{I} = \{\kappa_i, W_i, p_i, \dot{x}_i, \dot{A}_i, P_i\}_{i\leq n}$ be a rank progression of length n.

- By $\vec{I}\restriction k$ we mean the rank progression $\{\kappa_i, W_i, p_i, \dot{x}_i, \dot{A}_i, P_i\}_{i\leq k}$.
- Let $j \leq n$ be least so that $j \geq k$ or $\mathrm{lh}(P_j) > k$. (If no such j exists let $j = n$.) By $\vec{I}\|k$ we mean the rank progression $\{\kappa_i, W_i, p_i, \dot{x}_i, \dot{A}_i, P_i'\}_{i\leq j}$ where $P_i' = P_i$ for $i < j$ and $P_j' = P_j\restriction k$. (Note that $\max\{j, \mathrm{lh}(P_1'), \ldots, \mathrm{lh}(P_j')\} \leq k$. So if x fits $\vec{I}$, and x^* agrees with x to k, then x^* fits $\vec{I}\|k$.)

Let ζ be an ordinal, let x be a code, and let $I = \{\kappa_i, W_i, p_i, \dot{x}_i, \dot{A}_i, P_i\}_{i\leq n}$ be a rank progression. Let $\delta < \kappa$ be a Woodin cardinal of M.

We now work to define:

(A) A $\mathrm{col}(\omega, \delta)$–name $\dot{A}(\delta, \vec{I}, \zeta)$ for a set of codes;
(B) A meaning for the statement "$(x, \vec{I}, \zeta)$ is good for I"; and
(C) A game $G(x, \vec{I}, \zeta)$.

We make the definitions by simultaneous induction on ζ.

Definition 4.4. Define $\dot{A}(\delta, \vec{I}, \zeta)$ to be the canonical name in $\mathrm{col}(\omega, \delta)$ for the set of codes x so that: x fits $\vec{I}$, and player I has a winning strategy in $G(x, \vec{I}, \zeta)$.

Definition 4.5. Suppose that x fits $\vec{I}$. We define $G(x, \vec{I}, \zeta)$ under this assumption. Let l be the expectation at x. $G(x, \vec{I}, \zeta)$ starts with player I playing some ordinal $\zeta^* < \zeta$. The game continues according to one of the following cases.

Case 1. If $n < l$. In this case the players proceed according to Diagram 5. Player I plays a location $\langle p_{n+1}, \dot{x}_{n+1}\rangle$ in V_{κ_n} which is an n–extension of $\langle p_n, \dot{x}_n\rangle$, and so that p_{n+1} forces the value of $\dot{x}_{n+1}\restriction n+1$ to be precisely $x\restriction n+1$. I and II then play the basic rank game associated to κ_n, W_n, and $\langle p_{n+1}, \dot{x}_{n+1}\rangle$. II's moves give rise to κ_{n+1} and W_{n+1}.

At this point the game ends. Let δ_{n+1} be the first Woodin cardinal of M above κ_{n+1}. Let $\dot{A}_{n+1}$ be the name $\dot{A}(\delta_{n+1}, \vec{I}, \zeta^*)$. (This we can do inductively since $\zeta^* < \zeta$.) Let P_{n+1} be the empty position in $\mathcal{A}_{n+1}[x]$. Let $\vec{I}^*$ be the rank progression, of length $n+1$, which extends $\vec{I}$ with the objects (κ_{n+1}, W_{n+1}, etc.) determined above.

Player I wins the run of $G(x, \vec{I}, \zeta)$ described above just in case that $(x, \vec{I}^*, \zeta^*)$ is good for I. (Here again we apply induction, using the fact that $\zeta^* < \zeta$.) $\dashv$

Case 2. If $n \geq l$. In this case the game proceeds according to Diagram 6. Let $i = \mathrm{lh}(P_l)$. I and II play round i in $\mathcal{A}_l[x]$ following the position P_l. Let $P_l^* = P_l$—$, a_{i-\mathrm{I}}, a_{i-\mathrm{II}}$ be the position they generate. Let $\vec{I}'$ be the result of replacing P_l in $\vec{I}$ by P_l^* (and not changing anything else), and let $\vec{I}^* = \vec{I}'\restriction l$.

Player I wins the run of $G(x, \vec{I}, \zeta)$ described above just in case that $(x, \vec{I}^*, \zeta^*)$ is good for I. ⊣

$$\begin{array}{c|cccc}
\mathrm{I} & \zeta^* & \langle p_{n+1}, \dot{x}_{n+1}\rangle & U_{n+1} & \\
\hline
\mathrm{II} & & & & \kappa_{n+1}, W_{n+1}
\end{array}$$

Diagram 5. The game $G(x, \vec{I}, \zeta)$ if $n < l$.

$$\begin{array}{c|ccc}
\mathrm{I} & \zeta^* & a_{i-\mathrm{I}} & \\
\hline
\mathrm{II} & & & a_{i-\mathrm{II}}
\end{array}$$

Diagram 6. The game $G(x, \vec{I}, \zeta)$ if $n \geq l$.

Remark 4.6. Notice that l, the expectation at x, is not equal to 0 by Remark 1.8. Thus in case 2 we know that $0 < l \leq n$, and the reference to $\mathcal{A}_l$ and P_l is valid. (The rank progression $\vec{I}$ includes the objects $\dot{A}_i$ and P_i for $0 < i \leq n$, see Definition 4.2.)

Finally, we define the meaning of the statement $(x, \vec{I}, \zeta)$ is good for I. Let m meter x. Let $\bar{\kappa} < \kappa_m$. Let $\bar{\delta}$ be the first Woodin cardinal of M above $\bar{\kappa}$. Let $\dot{A}_{\bar{\kappa}}$ be $\dot{A}(\delta, \vec{I}\|m, \zeta)$.[1] Let $\mathcal{A}_{\bar{\kappa}}$ be the auxiliary games map for $\bar{\kappa}$ and $\dot{A}_{\bar{\kappa}}$.

Definition 4.7. Let $G^*(\bar{\kappa}, x, \vec{I}, \zeta)$ be the game in which players I and II collaborate to produce a real y, and at the same time play moves in $\mathcal{A}_{\bar{\kappa}}[x—, y]$. The first player to violate any of the rules of $\mathcal{A}_{\bar{\kappa}}[x—, y]$ loses. Infinite runs are won by player II. The format of the game is displayed in Diagram 7.

Note that in defining the game we use the continuity of the map $\mathcal{A}_{\bar{\kappa}}$: the rules for round n in $\mathcal{A}_{\bar{\kappa}}[x—, y]$ depend only on (x and) $y\restriction n+1$.

Definition 4.8. $(x, \vec{I}, \zeta)$ is **good for** I just in case that there are arbitrarily large $\bar{\kappa}$ below κ_m so that player I has a winning strategy in $G^*(\bar{\kappa}, x, \vec{I}, \zeta)$.

$$\begin{array}{c|cccccccc}
\mathrm{I} & y(0) & a_{0-\mathrm{I}} & & & a_{1-\mathrm{I}} & & \dots \\
\hline
\mathrm{II} & & & a_{0-\mathrm{II}} & y(1) & & a_{1-\mathrm{II}} &
\end{array}$$

Diagram 7. The game $G^*(\bar{\kappa}, x, \vec{I}, \zeta)$.

We have now completed the definitions indicated in items (A)–(C) above.

Recall that our goal is to define the tree D and the projection ρ. Let $\langle \zeta_{\mathrm{L}}, \zeta_{\mathrm{H}}\rangle$ be the lexicographically least (minimizing first over ζ_{H}) pair of local indiscernibles for κ in M, see Neeman [5, Definition 1A.15]. Let x_0 be the code for the empty

[1] $\vec{I}\|m$ is defined also if $\mathrm{lh}(\vec{I}) < m$, see Definition 4.3.

position. For each $W \subset \mathrm{V}_{\kappa+1}$ let $\vec{I}_W$ be the rank progression of length 0 determined by the assignment $W_0 = W$.

Definition 4.9. W is **good for** I just in case that $(x_0, \vec{I}_W, \zeta_{\mathrm{L}})$ is good for I.

Definition 4.10. Define D to be the tree

$$\bigcup_{W \text{ is good for I}} T_W,$$

and define $\rho\colon D \to \omega^{<\omega}$ to be

$$\bigcup_{W \text{ is good for I}} \pi_W.$$

(See Definitions 2.6 and 2.7 for the definitions of T_W and π_W.)

§5. A proof of Lemma 3.1. We work in this Section with the tree D and the projection ρ of the previous section. In addition to these objects let us fix Ψ, a strategy for I on D which is close to M. We work to prove that player I has a winning strategy in $\widehat{G}_{\mathrm{adm}}(\Psi, M, \kappa)$.

Let $\langle \zeta_{\mathrm{L}}, \zeta_{\mathrm{H}} \rangle$ be the least pair of local indiscernibles for κ in M.

Definition 5.1. Let x be a code. A **potential expanded rank progression** (**perp** for short) wrt x consists of sequences $\langle \kappa_i, W_i, p_i, \dot{x}_i, M_i, r_i, \zeta_i \mid i \leq n \rangle$ and $\langle \dot{A}_i, \mathfrak{P}_i, \mu_i \mid 0 < i \leq n \rangle$ satisfying conditions (1)–(8) below.

1. $M_n = M$;
2. Each r_i belongs to M_i;
3. $\zeta_0 = \zeta_{\mathrm{L}}$, and $\zeta_i < \zeta_{i-1}$ for $i > 0$;
4. μ_i is a measure in M_i with critical point κ_i;
5. $\dot{A}_i$ is a $\mathrm{col}(\omega, \delta_i)$–name for a set of codes in M_i^*, where $M_i^* = \mathrm{Ult}(M_i, \mu_i)$, and δ_i is the first Woodin cardinal of M_i^* above κ_i;
6. $\mathfrak{P}_i$ is a position in $\mathcal{A}_{i\text{-piv}}[x]$, played according to $\sigma_{i\text{-piv}}[x]$, where $\mathcal{A}_{i\text{-piv}}$ is the pivot games map associated to $\dot{A}_i$, δ_i, and $X = M_i^* \| \kappa_i$, and $\sigma_{i\text{-piv}}$ is the corresponding pivot strategies map (see Neeman [5, §1C] for the definitions) in M_i^*;
7. M_{i-1} is the final model given by the position $\mathfrak{P}_i$.

$\mathfrak{P}_i$ includes a finite iteration tree $\mathcal{T}_i$ on M_i^*, with its final model M_{i-1}. Let $h_i\colon M_i^* \to M_{i-1}$ be the embedding given by this tree. $\mathfrak{P}_i$ includes further a position in $h_i(\mathcal{A}_i)[x]$. Let P_i denote this position.

8. The sequence $\vec{I} = \{\kappa_i, W_i, p_i, \dot{x}_i, h_i(\dot{A}_i), P_i\}_{i \leq n}$ is a rank progression wrt x, in the sense of M_0.

For short we denote the perp by $\vec{E} = \{\kappa_i, W_i, p_i, \dot{x}_i, \dot{A}_i, \mathfrak{P}_i, M_i, \mu_i, r_i, \zeta_i\}_{i \leq n}$. We use $\vec{I}(\vec{E})$ to denote the rank progression in condition (8). We refer to n as the length of $\vec{E}$. By $\vec{E}|k$ we mean $\{\kappa_i, W_i, p_i, \dot{x}_i, \dot{A}_i, \mathfrak{P}_i, M_i, \mu_i, r_i, \zeta_i\}_{i \leq k}$. This is a perp over M_k.

Remark 5.2. Each of the maps $\sigma_{i\text{-piv}}$ above takes *two* arguments: the real x and a function $\varrho\colon \omega \to M_i^* \| \delta_i + 1$. ϱ must be *onto* $M_i^* \| \delta_i + 1$ for infinite runs according to $\sigma_{i\text{-piv}}[\varrho, x]$ to produce pivots, see Neeman [5, Lemma 1C.5]. We are suppressing the mention of ϱ completely here. Its choice is a matter of book-keeping, which the careful reader will fold into the constructions below. Let us only point out that it is in the choice of these maps that condition (C) at the start of Section 3 (stating that $M\|\kappa+1$ is countable in V) is used.

The potential expanded rank progression induces embeddings $j_{i,i-1}\colon M_i \to M_{i-1}$ (obtained by composing the ultrapower embedding by μ_i, to pass from M_i to M_i^*, with the embedding $h_i\colon M_i^* \to M_{i-1}$). By composition then we obtain embeddings $j_{k,l}\colon M_k \to M_l$ for $l \leq k \leq n$.

Definition 5.3. Let $\vec{E} = \{\kappa_i, W_i, p_i, \dot{x}_i, \dot{A}_i, \mathfrak{P}_i, M_i, \mu_i, r_i, \zeta_i\}_{i\leq n}$ be a perp over M, and let $\vec{I}$ be as in condition (8) of Definition 5.1. We make the following definitions:

- By $\mu(\vec{E})$ we mean μ_n, by $\delta(\vec{E})$ we mean δ_n, by $r(\vec{E})$ we mean r_n, by $\mathfrak{P}(\vec{E})$ we mean $\mathfrak{P}_n$, by $\mathcal{A}_{\text{piv}}(\vec{E})$ we mean $\mathcal{A}_{n\text{-piv}}$, and by $\zeta(\vec{E})$ we mean ζ_n.
- We say that $(x, \vec{E})$ is **good for** I just in case that $(x, \vec{I}, \zeta_n)$ is good for I in M_0.
- By $G(x, \vec{E})$ we mean the game $G(x, \vec{I}, \zeta_n)$, as computed in M_0.

Let μ be a measure on κ in M. Let $i_\mu\colon M \to \mathrm{Ult}(M,\mu)$ be the ultrapower embedding by μ. Let δ be the first Woodin cardinal of $\mathrm{Ult}(M,\mu)$ above κ. We make the following additional definitions.

- By $\dot{A}(\mu, \vec{E})$ we mean the name $\dot{A}(\delta, i_\mu(\vec{I}), i_\mu(\zeta_n))$, as computed in $i_\mu(M_0)$.
- By $G^*(x, \mu, \vec{E})$ we mean the game $G^*(\kappa, x, i_\mu(\vec{I}), i_\mu(\zeta_n))$, as computed in $i_\mu(M_0)$.

More generally, we use $G^*(x, \mu, \vec{E}, \tau)$ to denote $G^*(\tau, x, i_\mu(\vec{I}), i_\mu(\zeta_n))$, and use $\dot{A}(\mu, \vec{E}, \tau)$ to denote $\dot{A}(\delta_\tau, i_\mu(\vec{I}), i_\mu(\zeta_n))$ where δ_τ is the first Woodin cardinal of $i_\mu(M_0)$ above τ.

Definition 5.4. An **expanded rank progression** (**erp** for short) wrt x is a potential expanded rank progression $\vec{E}$ wrt x which satisfies the following conditions (in addition to the conditions in Definition 5.1) for each $i \leq n$:

1. κ_i is precisely equal to $j_{n,i}(\kappa)$;
2. r_i is a whole position in $j_{n,i}(D)$, played according to $j_{n,i}(\Psi)$;
3. The ending of r_i is precisely $\langle p_i, \dot{x}_i, W_i\rangle$;
4. r_i belongs to the range of $j_{n,i}$, and (if $i < n$) r_{i+1} strictly extends $j^{-1}_{i+1,i}(r_i)$;
5. $\vec{E}|i$ belongs to the range of $j_{n,i}$; and
6. (for $i > 0$) $\dot{A}_i$ is precisely equal to $\dot{A}(\delta_i, \vec{I}\restriction i-1, \zeta_{i-1})$ as computed over M_0, where δ_i is the first Woodin cardinal of M_0 above κ_i.

Definition 5.5. By $\vec{E}\restriction k$ we mean $j^{-1}_{n,k}(\vec{E}|k)$. (The pullback by $j_{n,k}$ makes sense using condition (5) in Definition 5.4.) We say that $\vec{E}^*$ **extends** $\vec{E}$ just in case that $\vec{E}^*\restriction \mathrm{lh}(\vec{E})$ is equal to $\vec{E}$. We say that $\vec{E}^*$ and $\vec{E}$ agree to k just in case that $\vec{E}^*\restriction k = \vec{E}\restriction k$.

Claim 5.6. *Let x be a code. Suppose that $\langle \vec{E}_n \mid n < \omega\rangle$ is a sequence of expanded rank progressions wrt x, so that $\mathrm{lh}(\vec{E}_n) = n$ and for each $n > 0$, $\vec{E}_n$*

extends $\vec{E}_{n-1}$. Then there exists an infinite branch $\vec{r}$ through D so that: $\vec{r}$ is according to Ψ, and $\rho(\vec{r}) = x$.

Proof. Let $\vec{r} = \bigcup_{n<\omega} r(\vec{E}_n)$. Chasing through the definitions we see that $\vec{r}$ is an infinite branch through D, played according to Ψ. From condition (3) in Definition 5.4 and condition (4) in Definition 4.2 it follows that $\rho(\vec{r}) = x$. $\dashv$

Definition 5.7. Let $\vec{E} = \{\kappa_i, W_i, p_i, \dot{x}_i, \dot{A}_i, \mathfrak{P}_i, M_i, \mu_i, r_i, \zeta_i\}_{i\leq n}$ be an expanded rank progression. By the n**th level** of $\vec{E}$ we mean the objects κ_n, $W_n, \ldots, \mathfrak{P}_n, \ldots, \zeta_n$. $\vec{E}$ thus consists of its nth level and the objects in $\vec{E}|n-1$. Recall that $\mathfrak{P}_n$ includes an iteration tree, leading to the model M_{n-1} and giving rise to the embedding $j_{n,n-1}$ defined above. We say that the nth level **leads** to M_{n-1} and $j_{n,n-1}$.

Definition 5.8. Let $\vec{E} = \{\kappa_i, W_i, p_i, \dot{x}_i, \dot{A}_i, \mathfrak{P}_i, M_i, \mu_i, r_i, \zeta_i\}_{i\leq n}$ be an expanded rank progression. Let $\mathfrak{P}^*$ be a position in $\mathcal{A}_{n\text{-piv}}$, extending $\mathfrak{P}_n$ by one round. Let $\mathcal{T}^*$ and $\mathcal{T}$ be the iteration tress in $\mathfrak{P}^*$ and $\mathfrak{P}_n$ respectively. $\mathcal{T}^*$ then extends $\mathcal{T}$. Let h be the embedding, given by $\mathcal{T}^*$, from the final even model of $\mathcal{T}$ to the final even model of $\mathcal{T}^*$. Let ζ^* be an ordinal smaller than $h(\zeta_n)$.

Define the **line extension** of $\vec{E}$ by $\mathfrak{P}^*$ and ζ^* to be the progression $\vec{E}^*$ determined by the conditions:

- The nth level of $\vec{E}^*$ consists of the objects in the nth level of $\vec{E}$, with $\mathfrak{P}_n$ replaced by $\mathfrak{P}^*$ and ζ_n replaced by ζ^*; and
- $\vec{E}^*|n-1$ is equal to $h(\vec{E}|n-1)$.

The shift by h in the last item is necessary: The nth line in $\vec{E}$ and the nth line in $\vec{E}^*$ lead to different models. h embeds the former into the latter.

Claim 5.9. *Let x be a code. Let $\vec{E}_k$ ($k<\omega$) be a sequence of expanded rank progressions wrt x, so that, for each $k>0$, $\vec{E}_k$ is a line extension of $\vec{E}_{k-1}$.*

Let $\mu = \mu(\vec{E}_0)$ and let $\delta = \delta(\vec{E}_0)$ (these are the same as $\mu(\vec{E}_k)$ and $\delta(\vec{E}_k)$ for each k). Let $n = \mathrm{lh}(\vec{E}_0)$ (this is the same as $\mathrm{lh}(\vec{E}_k)$ for each k). Let $\vec{F} = \vec{E}_0\restriction n-1$ (this is the same as $\vec{E}_k\restriction n-1$ for each k).

Then there is a length ω iteration tree $\mathcal{T}$ on $\mathrm{Ult}(M,\mu)$, so that for every wellfounded cofinal branch b through $\mathcal{T}$:

- *Player I has a winning strategy in $G(x, (j_b\circ i_\mu)(\vec{F}))$ (as defined over M_b),*

where M_b and j_b are the direct limit model and embedding along b.

Proof. Let $\mathfrak{P}_\infty = \bigcup_{k<\omega}\mathfrak{P}(\vec{E}_k)$. $\mathfrak{P}_\infty$ is then an $\mathcal{A}_n$–pivot for x over $\mathrm{Ult}(M,\mu)$. (See [5, Lemma 1C.5]. It is here that Remark 5.2 comes into play. We continue to suppress the function ϱ involved.)

Let $\mathcal{T}$ be the iteration tree given by $\mathfrak{P}_\infty$.

Suppose b is a cofinal wellfounded branch through $\mathcal{T}$. The ordinals $\zeta(\vec{E}_k)$ witness that the even branch of $\mathcal{T}$ is illfounded. (This follows from the requirement $\zeta^* < h(\zeta_n)$ in Definition 5.8.) So b must be an odd branch.

Since $\mathfrak{P}_\infty$ is a $\mathcal{A}_n$–pivot for x, we know that there is a $\mathrm{col}(\omega,\delta)$–generic h over $\mathrm{Ult}(M,\mu)$ so that:

- x belongs to $j_b(\dot{A}(\mu,\vec{F}))[h]$.

(We already folded into this the fact that $\dot{A}_n$ is equal to $\dot{A}(\mu, \vec{F})$, given by condition (6) in Definition 5.4 and the definition of $\vec{F}$ here.) If follows from this, Definition 5.3, and Definition 4.4, that player I has a winning strategy in $G(x, (j_b \circ i_\mu)(\vec{F}))$. $\dashv$

Definition 5.10. Let $\vec{E} = \{\kappa_i, W_i, p_i, \dot{x}_i, \dot{A}_i, \mathfrak{P}_i, M_i, \mu_i, r_i, \zeta_i\}_{i \leq n}$ be an expanded rank progression. Let $m < \omega$ be given. Let $j \leq n$ be least so that $j \geq m$ or $\mathrm{lh}(\mathfrak{P}_j) > m$. (If no such j exists then let $j = n$.) Define $\vec{E} \| m$ to be the expanded rank progression obtained from $\vec{E} \restriction j$ by further restricting $\mathfrak{P}_j$ to $\mathfrak{P}_j \restriction m$.

Recall that our goal here is to produce a winning strategy for player I in the game $\widehat{G}_{\mathrm{adm}}(\Psi, M, \kappa)$. Fix an imaginary opponent willing to play for II. We describe how to play against this opponent, and win. The description as usual takes the form of a construction, joint with the opponent, of a run of the game. We construct in mega-rounds. At the start of mega-round α, for α a successor ordinal, we have:

(A) A position $\langle y_\xi \mid \xi < \alpha - 1\rangle$ in G_{adm}. Let $x^{\alpha-1}$ code this position.
(B) An iterate M^α of $M = M^1$ and an iteration embedding $j^{1,\alpha} \colon M \to M^\alpha$.
(C) An expanded rank progression $\vec{E}^\alpha$, over M^α, wrt $x^{\alpha-1}$ if α is a successor, and wrt x^α if α is a limit.

Inductively we make sure that:

(i) $x^{\alpha-1}$ belongs to a generic extension of M^α by a poset of size less than $\kappa^\alpha = j^{1,\alpha}(\kappa)$.
(ii) $(x^{\alpha-1}, \vec{E}^\alpha)$ is good for I, over M^α.
(iii) Let λ be a limit. Let $k < \omega$. Suppose that for all sufficiently large $\alpha < \lambda$, the expectation at $\langle y_\xi \mid \xi < \alpha\rangle$ is larger than k. Then for all sufficiently large $\alpha < \lambda$, $\vec{E}^\lambda$ and $j^{\alpha,\lambda}(\vec{E}^\alpha)$ agree to k.

Let us now begin the construction for mega-round α. Let $m = m^{\alpha-1}$ meter $x^{\alpha-1}$. Let μ^α be some measure on κ^α in M^α.

From the fact that $(x^{\alpha-1}, \vec{E}^\alpha)$ is good for I it follows that, for some τ between κ^α and $i_{\mu^\alpha}(\kappa^\alpha)$, player I has a winning strategy in $G^*(x^{\alpha-1}, \mu^\alpha, \vec{E}^\alpha, \tau)$. (In fact there are arbitrarily large such τ below $i_{\mu^\alpha}(\kappa^\alpha)$.) By condition (i) the game belongs to a small (relative to κ^α) generic extension of $\mathrm{Ult}(M^\alpha, \mu^\alpha)$ and since the game is open we can find a winning strategy for I in this extension. Using this strategy, a construction (joint with the imaginary opponent) of the kind done in [5] (for example in Section 1E and in Chapter 2) produces a real $y_{\alpha-1}$, an iteration tree $\mathcal{T}^\alpha$ on $\mathrm{Ult}(M^\alpha, \mu^\alpha)$, and a cofinal branch b^α through the tree $\mathcal{T}^\alpha$, so that, letting h^α be the direct limit embedding along b^α and setting $x^\alpha = x^{\alpha-1}—, y_{\alpha-1}$, we get:

(iv) There is a generic g^α so that x^α belongs to $h^\alpha(\dot{A}(\mu^\alpha, \vec{E}^\alpha \| m, \tau))[g^\alpha]$.

Let $\vec{F}^{\alpha+1} = j^{\alpha,\alpha+1}(\vec{E}^\alpha \| m)$. Using the definitions in Section 3 it follows from condition (iv) that:

(v) Player I has a winning strategy in $G(x^\alpha, \vec{F}^{\alpha+1})$, as computed over $M^{\alpha+1}$.

Let l be the expectation at x^α. Let $n = n^\alpha = \mathrm{lh}(\vec{F}^{\alpha+1})$. Note that:

(vi) n^α is either equal to $\mathrm{lh}(\vec{E}^\alpha)$, or else it is least so that $n^\alpha \geq m^{\alpha-1}$ or $\mathrm{lh}(P(\vec{E}^\alpha \restriction n^\alpha)) > m^{\alpha-1}$.

We now divide the construction into two cases, depending on whether $l > n$ or $l \leq n$. (These two cases correspond to the two cases in Definition 4.5.)

Case 1. If $n < l$. $G(x^\alpha, \vec{F}^{\alpha+1})$ is then played according to the rules of case 1 in Definition 4.5. I's strategy in the game, and the strategy $j^{1,\alpha+1}(\Psi)$ for I in $j^{1,\alpha+1}(D)$, combine through Lemma 2.8 to produce ζ_{n+1}, p_{n+1}, $\dot{x}_{n+1}$, U_{n+1}, μ_{n+1}, and W_{n+1}. (These are all the objects involved in moves in $G(x^\alpha, \vec{F}^{\alpha+1})$ and moves in $j^{1,\alpha+1}(D)$ following the position $r(\vec{F}^{\alpha+1})$.) These objects (and the assignment $P_{n+1} = \emptyset$) define an expanded rank progression, of length $n^\alpha + 1$, extending $\vec{F}^{\alpha+1}$. Let $\vec{E}^{\alpha+1}$ be this progression. We have:

(a) $\vec{E}^{\alpha+1}$ extends $j^{\alpha,\alpha+1}(\vec{E}^\alpha \| m)$, where m meters x^α.

Moreover, the winning condition in Definition 4.5 is such that:

(b) $(x^\alpha, \vec{E}^{\alpha+1})$ is good for I, over $M^{\alpha+1}$.

We can now pass to mega-round $\alpha + 1$. ⊣ (Case 1)

Case 2. If $n \geq l$. $G(x^\alpha, \vec{F}^{\alpha+1})$ is then played according to the rules of case 2 in Definition 4.5. I's strategy in the game combines with the pivot strategies map corresponding to $\mathcal{A}_l$ to precisely produce a line-extension of $\vec{F}^{\alpha+1} \restriction l$. Let $\vec{E}^{\alpha+1}$ be this extension. For the record let us note then that:

(c) $\vec{E}^{\alpha+1}$ is a line extension of $j^{\alpha,\alpha+1}((\vec{E}^\alpha \| m) \restriction l)$.

The payoff in case 2 of Definition 4.5 is such that:

(d) $(x^\alpha, \vec{E}^{\alpha+1})$ is good for I, over $M^{\alpha+1}$.

This allows us to pass to mega-round $\alpha + 1$. ⊣ (Case 2)

For the record let us note that:

(e) $\mathfrak{P}_j(\vec{E}^{\alpha+1})$ can be longer than $\mathfrak{P}_j(\vec{E}^\alpha)$ only if j is expected at x^α.

Suppose now that our construction has reached a limit mega-round λ. We have, through the work in the previous mega-rounds, a position $\langle y_\xi \mid \xi < \lambda \rangle$, and an iteration $\langle M^\xi, j^{\zeta,\xi} \mid 1 \leq \zeta \leq \xi < \lambda \rangle$. Let M^λ be the direct limit of this iteration, and let $j^{\zeta,\lambda}$ for $\zeta < \lambda$ be the direct limit embeddings. Let x^λ code $\langle y_\xi \mid \xi < \lambda \rangle$.

The position $\langle y_\xi \mid \xi < \lambda \rangle$ may, or may not, be terminal in G_{adm}. Suppose first that it is *not* terminal, and let l meter $\langle y_\xi \mid \xi < \lambda \rangle$. l is then the least number which is expected at x^α for cofinally many $\alpha < \lambda$.

Let m^α be the number which meters x^α. Using conditions (iii), (vi), (a), (c), and (e), using the fact that l is the least number expected at x^α for cofinally many $\alpha < \lambda$, and using the fact that $m^\alpha \to \infty$ as $\alpha \to \lambda$, we can find an increasing sequence $\langle \alpha_n \mid n < \omega \rangle$ of successor ordinals, cofinal in λ, so that:

- $\mathrm{lh}(\vec{E}^{\alpha_n}) = l$ for each n; and
- for each $n > 0$, $\vec{E}^{\alpha_n}$ is a line-extension of $j^{\alpha_{n-1},\alpha_n}(\vec{E}^{\alpha_{n-1}})$.

Moreover we can arrange that $x^{\alpha_n - 1}$ and x^λ are in sufficient agreement that $\vec{E}^{\alpha_n}$ is an expanded rank progression wrt x^λ. (Notice that the codes $x^{\alpha_n - 1}$ converge to x^λ, by Claim 1.13.)

Let $\vec{E}^*_n = j^{\alpha_n,\lambda}(\vec{E}^{\alpha_n})$. Then each $\vec{E}^*_n$ is an expanded rank progression wrt x^λ over M_λ, and for each $n > 0$, $\vec{E}^*_n$ is a line extension of $\vec{E}^*_{n-1}$. We are therefore in a position to apply Claim 5.9.

Let $\mu^\lambda = \mu(\vec{E}^*_0)$. Let $\mathcal{T}^\lambda$ be the iteration tree given by Claim 5.9. We play these objects for I in mega-round λ. Let b^λ be the response given by the imaginary opponent (playing for II). Let $M^{\lambda+1}$ be the direct limit along b^λ, and let j_{b^λ} be the direct limit embedding. Set $j^{\lambda,\lambda+1} = j_{b^\lambda} \circ i_{\mu^\lambda}$.

Let $\vec{E}^\lambda = \vec{E}^*_0 \restriction l-1$ (this is the same as $\vec{E}^*_n \restriction l-1$ for each n). This assignment satisfies condition (iii) above. This can be seen using the same condition below λ, using conditions (vi), (a), (c), and (e), using the fact that $m^\alpha \to \infty$ as $\alpha \to \lambda$, and using the fact that l is the least number expected cofinally often below λ.

If $M^{\lambda+1}$ is illfounded then player II loses $\widehat{G}_{\mathrm{adm}}$, and our job is done. So we may assume that b^λ leads to a wellfounded direct limit. By Claim 5.9 then:

- Player I has a winning strategy in $G(x^\lambda, j^{\lambda,\lambda+1}(\vec{E}^\lambda))$, as computed in $M^{\lambda+1}$.

Let $\vec{F}^{\lambda+1} = j^{\lambda,\lambda+1}(\vec{E}^\lambda)$. We can now continue following the construction in the successor mega-round, from condition (v) onward.

The description above handles the construction of a limit mega-round λ in the case that $\langle y_\xi \mid \xi < \lambda\rangle$ is not terminal in G_{adm}. Suppose now that $\langle y_\xi \mid \xi < \lambda\rangle$ *is* terminal. $\widehat{G}_{\mathrm{adm}}(\Psi, M, \kappa)$ ends at this point. It remains to check that it ends with a victory for player I.

For each $\alpha < \lambda$ let m^α meter $\langle y_\xi \mid \xi < \alpha\rangle$ and let l^α be the expectation at $\langle y_\xi \mid \xi < \alpha\rangle$. From the fact that $\langle y_\xi \mid \xi < \lambda\rangle$ is terminal it follows that *both* $m^\alpha \to \infty$ and $l^\alpha \to \infty$ as $\alpha \to \lambda$. Using conditions (iii), (vi), (a), (c), and (e) above it follows that for each $n < \omega$, there is $\alpha < \lambda$, so that $j^{\alpha,\eta}(\vec{E}^\alpha)$ and $\vec{E}^\eta$ agree to n for all $\eta \in [\alpha, \lambda)$. Let α_n denote the least such n. Let $\vec{E}_n = j^{\alpha_n,\lambda}(\vec{E}^{\alpha_n} \restriction n)$. Then each $\vec{E}_n$ is an expanded rank progression for x^λ over M^λ (we are using here the fact that $x^\alpha \to x^\lambda$ as $\alpha \to \lambda$), and $\vec{E}_n$ extends $\vec{E}_{n-1}$ for each $n > 0$. Using Claim 5.6 it follows that there exists an infinite branch $\vec{r}$ through $j^{1,\lambda}(D)$, so that $\vec{r}$ is according to $j^{1,\lambda}(\Psi)$, and so that $j^{1,\lambda}(\rho)(\vec{r}) = x$. This precisely is the winning condition for player I in $\widehat{G}_{\mathrm{adm}}(\Psi, M, \kappa)$.

Working with the model M, a cardinal κ in M satisfying the assumptions in Section 3, the tree D of Section 3, and a strategy Ψ for I on this tree, we described how to play for player I in the game $\widehat{G}_{\mathrm{adm}}(\Psi, M, \kappa)$, and win. This proves Lemma 3.1.

§6. E, χ, and Lemma 3.2. The definition of E, the definition of χ, and the proof of Lemma 3.2 precisely mirror Sections 4 and 5. Here we sketch the notation involved in the definitions of E and χ. (We shall use this notation later on, in Section 7.)

Let M and κ satisfy the assumptions in Section 3. We now mirror the definitions in Sections 2 and 4.

DEFINITION 6.1. Let $Z \subset \mathrm{V}_{\kappa+1}$, let $\langle p, \dot{x}\rangle$ be a location. The **mirrored basic rank game** associated to κ, Z, and $\langle p, \dot{x}\rangle$ is played according to the rules in Definition 2.4, except that the roles of the players are reversed (and we use V,

Z, and $\bar{Z}$ instead of U, W, and $\bar{W}$ in the notation). The format of the mirrored game is presented in Diagram 8.

$$\begin{array}{c|ll} \text{I} & & \langle \bar{\kappa}, \bar{Z} \rangle \\ \hline \text{II} & V & \end{array}$$

Diagram 8. The mirrored basic rank game.

Definition 6.2. Let $Z \subset \mathrm{V}_{\kappa+1}$. The **mirrored inverted rank game** associated to κ and Z is played according to the rules in Definition 2.5, except that again the roles of the players are reversed (and we use V, Z, and Z_n instead of U, W, and W_n in the notation). The format of the mirrored game is presented in Diagram 9.

$$\begin{array}{c|llll} \text{I} & \cdots\cdots\cdots & p_n, \dot{x}_n, V_n & & \cdots\cdots \\ \hline \text{II} & & & \mu_n, Z_n & \end{array}$$

Diagram 9. Round n in the inverted rank game.

Given $Z \subset \mathrm{V}_{\kappa+1}$ we use U_Z to denote the tree of the mirrored inverted rank game associated to κ and Z. We use $\sigma_Z \colon T_Z \to \omega^{<\omega}$ to denote the natural projection (mirroring Definition 2.7).

Let $\bar{\kappa} < \kappa$. Let δ be the first Woodin cardinal of M above $\bar{\kappa}$. For expository simplicity let g be $\mathrm{col}(\omega, \delta)$–generic/$M$. Let $\dot{B}$ be a $\mathrm{col}(\omega, \delta)$–name for a set of codes in $M[g]$. Let $\dot{B}'$ be the canonical name for $(M \| \delta)^\omega \times \dot{B}[g]$. Let $X = M \| \bar{\kappa}$. By the **mirrored auxiliary games map for $\bar{\kappa}$ and $\dot{B}$** we mean the mirrored auxiliary games map associated to $\dot{B}'$, δ, and X, as defined in Neeman [5, §1D].

Definition 6.3. A **mirrored rank progression** wrt x consists of finite sequences $\langle \kappa_i, Z_i, p_i, \dot{x}_i \mid i \leq n \rangle$ and $\langle \dot{B}_i, Q_i \mid 0 < i \leq n \rangle$ satisfying the conditions in Definition 4.2 (with Z_i, $\dot{B}_i$, and Q_i replacing W_i, $\dot{A}_i$, and P_i respectively), except that now Q_i is a position in $\mathcal{B}_i[x]$, where $\mathcal{B}_i$ is the *mirrored* auxiliary games map for κ_i and $\dot{B}_i$.

We write $\vec{J} = \{\kappa_i, Z_i, p_i, \dot{x}_i, \dot{B}_i, Q_i\}_{i \leq n}$ to refer to the mirrored progression. We define $\vec{J} \restriction k$ and $\vec{J} \| k$ as in Section 4.

Let ζ be an ordinal, let x be a code, and let $\vec{J} = \{\kappa_i, Z_i, p_i, \dot{x}_i, \dot{B}_i, Q_i\}_{i \leq n}$ be a mirrored rank progression. Let $\delta < \kappa$ be a Woodin cardinal of M. We work to define:

(A) A $\mathrm{col}(\omega, \delta)$–name $\dot{B}(\delta, \vec{J}, \zeta)$ for a set of codes;
(B) A meaning for the statement "$(x, \vec{J}, \zeta)$ is good for II"; and
(C) A game $H(x, \vec{J}, \zeta)$.

The definitions are by induction on ζ, following Section 4, only with the roles of the players reversed in the auxiliary moves:

Definition 6.4. $\dot{B}(\delta, \vec{J}, \zeta)$ is the canonical name in $\mathrm{col}(\omega, \delta)$ for the set of codes x so that x fits $\vec{J}$ (this notion is defined as in Section 4) and player II has a winning strategy in $H(x, \vec{J}, \zeta)$.

Definition 6.5. Suppose that x fits $\vec{J}$. Define $H(x, \vec{J}, \zeta)$ to be played according to the rules (of the two cases) in Definition 4.5, only with the roles of the players reversed, the basic rank game (in case 1) replaced by the mirrored basic rank game, and the map $\mathcal{A}_l$ (in case 2) replaced by the mirrored auxiliary games map $\mathcal{B}_l$. Diagrams 10 and 11 present the format of $H(x, \vec{J}, \zeta)$ in the two cases.

$$\begin{array}{c|ccccc}
\mathrm{I} & & & & & \kappa_{n+1}, Z_{n+1} \\
\hline
\mathrm{II} & \zeta^* & \langle p_{n+1}, \dot{x}_{n+1} \rangle & V_{n+1} & &
\end{array}$$

Diagram 10. The game $H(x, \vec{J}, \zeta)$ if $\mathrm{lh}(\vec{J})$ is smaller than the expectation at x (case 1).

$$\begin{array}{c|ccc}
\mathrm{I} & & & b_{i-\mathrm{II}} \\
\hline
\mathrm{II} & \zeta^* & b_{i-\mathrm{I}} &
\end{array}$$

Diagram 11. The game $H(x, \vec{J}, \zeta)$ if $\mathrm{lh}(\vec{J})$ is greater than or equal to the expectation at x (case 2).

Next we define the meaning of the statement $(x, \vec{J}, \zeta)$ is good for II. Let m meter x. Let $\bar{\kappa} < \kappa_m$. Let $\bar{\delta}$ be the first Woodin cardinal of M above $\bar{\kappa}$. Let $\dot{B}_{\bar{\kappa}}$ be $\dot{B}(\delta, \vec{J} \| m, \zeta)$. Let $\mathcal{B}_{\bar{\kappa}}$ be the mirrored auxiliary games map for $\bar{\kappa}$ and $\dot{B}_{\bar{\kappa}}$.

Definition 6.6. Define $H^*(\bar{\kappa}, x, \vec{J}, \zeta)$ to be the game in which players I and II collaborate to produce a real y, and at the same time play moves in $\mathcal{B}_{\bar{\kappa}}[x—, y]$. The first player to violate any of the rules of $\mathcal{B}_{\bar{\kappa}}[x—, y]$ loses. Infinite runs are won by player I. The format of the game is displayed in Diagram 12. (The definition here precisely mirrors Definition 4.7. Note that here the mirrored auxiliary games map is used, and infinite runs are won by I.)

Definition 6.7. $(x, \vec{J}, \zeta)$ is **good for** II just in case that there are arbitrarily large $\bar{\kappa}$ below κ_m so that player II has a winning strategy in $H^*(\bar{\kappa}, x, \vec{J}, \zeta)$.

$$\begin{array}{c|ccccccc}
\mathrm{I} & y(0) & & b_{0-\mathrm{I}} & & & b_{1-\mathrm{I}} & \\
\hline
\mathrm{II} & & b_{0-\mathrm{II}} & & y(1) & b_{1-\mathrm{II}} & & \ldots
\end{array}$$

Diagram 12. The game $H^*(\bar{\kappa}, x, \vec{I}, \zeta)$.

Let $\langle \zeta_{\mathrm{L}}, \zeta_{\mathrm{H}} \rangle$ be the lexicographically least pair of local indiscernibles for κ in M. Let x_0 be the code for the empty position. For each $Z \subset \mathrm{V}_{\kappa+1}$ let $\vec{J}_Z$ be the mirrored rank progression of length 0 determined by the assignment $Z_0 = Z$. Define finally:

- Z is **good for** II just in case that $(x_0, \vec{J}_Z, \zeta_{\text{L}})$ is good for II,
- E is the tree
$$\bigcup_{Z \text{ is good for II}} U_Z,$$
- $\chi\colon E \to \omega^{<\omega}$ is the map
$$\bigcup_{Z \text{ is good for II}} \sigma_Z.$$

These definitions precisely mirror the definitions of D and ρ in Section 4.

Lemma 3.2 can now be proved through an argument which precisely mirrors the proof of Lemma 3.1 in Section 5. We omit further details.

§7. Lemma 3.3. We work in this section to prove Lemma 3.3. Fix M and κ satisfying conditions (A)–(C) of Section 3. Let D, E, ρ, and χ be the trees and projections of Sections 4 and 6. Let Ψ be a strategy for II on D and let Ω be a strategy for I on E. We have to find branches $\vec{d} \in [D]$ according to Ψ, and $\vec{e} \in [E]$ according to Ω, so that $\rho(\vec{d}) = \chi(\vec{e})$.

We begin with definitions very much in the spirit of Section 5. (We use the same terms, perp and expanded rank progression, though the definitions here differ from the ones of Section 5 in some respects.) We then present the construction of $\vec{d}$ and $\vec{e}$.

Fix some h which is $\text{col}(\omega, {<}\kappa)$–generic over M.

Definition 7.1. Let x be a code. A **potential expanded rank progression** (**perp** for short) with respect to x and $\langle g_i \mid 0 < i \leq n\rangle$ consists of sequences $\langle \kappa_i, W_i, Z_i, p_i, \dot{x}_i, M_i, r_i, s_i \mid i \leq n\rangle$ and $\langle \dot{A}_i, \dot{B}_i, P_i, Q_i, \mu_i \mid 0 < i \leq n\rangle$ satisfying conditions (1)–(11) below.

1. $M_n = M$;
2. r_i and s_i belong to M_i;
3. $\langle p_i, \dot{x}_i\rangle$ is a location (the empty location if $i = 0$);
4. μ_i is a measure on κ_i in M_i;
5. $M_{i-1} = \text{Ult}(M_i, \mu_i)$. We also use M_i^* to denote $\text{Ult}(M_i, \mu_i)$;
6. g_i is $\text{col}(\omega, \delta_i)$–generic over M_i^*, where δ_i is the first Woodin cardinal of M_i^* above κ_i;
7. $\dot{A}_i$ and $\dot{B}_i$ are $\text{col}(\omega, \delta_i)$–names for sets of codes in M_i^*;
8. P_i is a position in $\mathcal{A}_i[x]$, played according to $\sigma_{i\text{-gen}}[x]$, where $\mathcal{A}_i$ is the auxiliary games map associated to $(M\|\delta_i) \times \dot{A}_i$, δ_i, and $X_i = M_i^*\|\kappa_i$ over M_i^*, and $\sigma_{i\text{-gen}}$ is the corresponding generic strategies map (defined relative to the generic g_i);
9. Q_i is a position in $\mathcal{B}_i[x]$, played according to $\tau_{i\text{-gen}}[x]$, where $\mathcal{B}_i$ is the mirrored auxiliary games map associated to $(M\|\delta_i) \times \dot{B}_i$, δ_i, and $X_i = M_i^*\|\kappa_i$ over M_i^*, and $\tau_{i\text{-gen}}$ is the corresponding mirrored generic strategies map (defined relative to the generic g_i);

10. The sequence $\vec{I} = \{\kappa_i, W_i, p_i, \dot{x}_i, \dot{A}_i, P_i\}_{i \leq n}$ is a rank progression wrt x, in the sense of M_0; and
11. The sequence $\vec{J} = \{\kappa_i, Z_i, p_i, \dot{x}_i, \dot{B}_i, Q_i\}_{i \leq n}$ is a mirrored rank progression wrt x, in the sense of M_0

We use $\vec{E}$ to denote perps, refer to n as the length of the perp, and define $\vec{E}|k$ to be the perp, over M_k, consisting of the objects corresponding to $i \leq k$. We say that $\vec{E}$ is a perp wrt x if there exists some sequence $\langle g_i \mid 0 < i \leq n\rangle$ so that $\vec{E}$ is a perp wrt x and $\langle g_i \mid 0 < i \leq n\rangle$.

The dependence of Definition 7.1 on the sequence of generics $\langle g_i \mid 0 < i \leq n\rangle$ comes in through conditions (8) and (9). We sometimes have to emphasize the dependence on g_n, and in this case we say that $\vec{E}$ is a g_n–perp wrt x. We follow similar notation in connection with Definition 7.2 below.

We use $\vec{I}(\vec{E})$ and $\vec{J}(\vec{E})$ to denote the progressions in conditions (10) and (11). We use $j_i\colon M_i \to M_{i-1}$ to denote the ultrapower embedding by μ_i, and use $j_{k,l}\colon M_k \to M_l$ for $l \leq k$ to denote the embeddings obtained from the j_is through composition.

We follow the terminology of Definition 5.3, adapted to the current settings. For example we use $\mu(\vec{E})$ to denote μ_n, use $P(\vec{E})$ to denote P_n, etc.

We say that $(x, \vec{E})$ is **good for** I just in case that $(x, \vec{I}(\vec{E}), \zeta_{\mathrm{L}})$ is good for I in M_0. ζ_{L} here is the lower ordinal in the lexicographically least pair of local indiscernibles for κ in M. We define **good for** II similarly, using $\vec{J}$.

By $G(x, \vec{E})$ we mean the game $G(x, \vec{I}(\vec{E}), \zeta_{\mathrm{L}})$, as computed in M_0. We define $H(x, \vec{E})$ similarly, using $\vec{J}$.

The definitions so far are adaptations of the ones in Definition 5.3. We adapt the last two clauses of that definition similarly, to define $\dot{A}(\mu, \vec{E})$, $\dot{B}(\mu, \vec{E})$, $G^*(x, \mu, \vec{E})$, and $H^*(x, \mu, \vec{E})$.

Definition 7.2. An **expanded rank progression** (**erp** for short) wrt x is a potential expanded rank progression $\vec{E}$ wrt x which satisfies the following conditions (in addition to the conditions in Definition 7.1) for each $i \leq n$:

1. κ_i is precisely equal to $j_{n,i}(\kappa)$;
2. r_i is a medial position (the empty position if $i = 0$) in $j_{n,i}(D)$, played according to $j_{n,i}(\Psi)$, and similarly s_i is a medial position (the empty position if $i = 0$) in $j_{n,i}(E)$, played according to $j_{n,i}(\Omega)$;
3. The end location of r_i is equal to $\langle p_i, \dot{x}_i\rangle$, and similarly with s_i;
4. $p_i \in h$, and x is an i–extension of $\dot{x}_i[h]$;
5. unrch(r_i) (as computed over M_i, relative to the strategy $j_{n,i}(\Psi)$) is precisely W_i, and similarly unrch(s_i) is precisely Z_i;
6. r_i and s_i belong to the range of $j_{n,i}$, and (if $i < n$) r_{i+1} and s_{i+1} strictly extend $j^{-1}_{i+1,i}(r_i)$ and $j^{-1}_{i+1,i}(s_i)$ respectively;
7. $\vec{E}|i$ belongs to the range of $j_{n,i}$; and
8. (for $i > 0$) $\dot{A}_i$ is precisely equal to $\dot{A}(\delta_i, \vec{I}\restriction i-1, \zeta_{\mathrm{L}})$ as computed over M_0, where δ_i is the first Woodin cardinal of M_0 above κ_i, and similarly $\dot{B}_i$ is precisely equal to $\dot{B}(\delta_i, \vec{I}\restriction i-1, \zeta_{\mathrm{L}})$.

By $\vec{E}\restriction k$ we mean $j^{-1}_{n,k}(\vec{E}|k)$. (The pullback by $j_{n,k}$ makes sense using condition (7) in Definition 7.2.) We say that $\vec{E}^*$ **extends** $\vec{E}$ just in case that $\vec{E}^*\restriction \mathrm{lh}(\vec{E})$ is equal to $\vec{E}$. We say that $\vec{E}^*$ and $\vec{E}$ agree to k just in case that $\vec{E}^*\restriction k = \vec{E}\restriction k$.

Remark 7.3. There are some structural differences between the definitions here and those in Section 5. For example here we deal with the actual auxiliary games map rather than the pivot games map, and with the generic strategies maps rather than the pivot games maps. Apart from these structural differences the key change is in conditions (2) and (5) of Definition 7.2, where instead of using whole positions we use medial positions, and instead of talking about the ending of r_i we talk about $\mathrm{unrch}(r_i)$.

Claim 7.4. *Suppose that $\langle \vec{E}_n \mid n < \omega\rangle$ is a sequence of expanded rank progressions, so that for each $n > 0$, $\vec{E}_n$ strictly extends $\vec{E}_{n-1}$. Then there are infinite branches $\vec{d} \in [D]$ and $\vec{e} \in [E]$, according to Ψ and Ω respectively, so that $\rho(\vec{d}) = x = \rho(\vec{e})$.*

Proof. Let $\vec{d} = \bigcup_{n<\omega} r(\vec{E}_n)$, and let $\vec{e} = \bigcup_{n<\omega} s(\vec{E}_n)$. Chasing through the definitions it is easy to check that $\vec{d} \in [D]$ is played according to Ψ, $\vec{e} \in [E]$ is played according to Ω, and $\rho(\vec{d}) = \chi(\vec{e})$. ⊣

Definition 7.5. Let $\vec{E}$ and $\vec{E}^*$ be expanded rank progressions of the same length n. $\vec{E}^*$ is a **line extension** of $\vec{E}$ just in case that: P^*_n strictly extends P_n; Q^*_n strictly extends Q_n; and other than this the two progressions are the same.

Claim 7.6. *Let $\vec{E}_0$ be an expanded rank progression of length $n+1$ wrt a code x_0. Let g be $\mathrm{col}(\omega, \delta(\vec{E}))$–generic over $\mathrm{Ult}(M, \mu(\vec{E}))$, so that $\vec{E}_0$ is a g-erp wrt x_0. Suppose that $\langle x_k, \vec{E}_k \mid 0 < k < \omega\rangle$ are such that for each $k < \omega$:*

1. *x_{k+1} is a code which extends x_k;*
2. *$\vec{E}_{k+1}$ is a g-erp wrt x_{k+1}; and*
3. *$\vec{E}_{k+1}$ is a line extension of $\vec{E}_k$.*

Let x_∞ code the union of the sequences coded by x_k, $k < \omega$. Suppose finally that $\langle x_k, \vec{E}_k \mid k < \omega\rangle$ belongs to $\mathrm{Ult}(M, \mu(\vec{E}_0))[g]$.

Let $\vec{E} = \vec{E}_0|n$. Then player II *has a winning strategy in $G(x, \vec{E})$, and player* I *has a winning strategy in $H(x, \vec{E})$.*

Proof. Let $\dot{A}$ denote $\dot{A}(\vec{E}_0)$ and let $\mathcal{A}$ denote $\mathcal{A}(\vec{E}_0)$.

By condition (3), $P(\vec{E}_{k+1})$ strictly extends $P(\vec{E}_k)$. Let $\vec{P} = \bigcup_{k<\omega} P(\vec{E}_k)$. $\vec{P}$ is then a generic run of $\mathcal{A}$. (This uses condition (8) of Definition 7.1.) Using Lemma 1B.2 of [5] it follows that $x \notin \dot{A}[g]$. By condition (8) of Definition 7.2 it follows that $x \notin \dot{A}(\delta(\vec{E}_0), \vec{I}(\vec{E}_0)\restriction n, \zeta_{\mathrm{L}})[g]$. By Definition 4.4 (and since $x \in M[g]$) this means that I does not have a winning strategy in $G(x, \vec{E}_0|n) = G(x, \vec{E})$. The game is finite, and hence determined, so II must have a winning strategy.

A similar argument using the positions $Q(\vec{E}_k)$ shows that I has a winning strategy in $H(x, \vec{E})$. ⊣

Claim 7.7. *Let $\vec{E}$ be an expanded rank progression leading to models M_i for $i \leq n$ and embeddings $j_{k,l}\colon M_k \to M_l$ for $l \leq k \leq n$. Then for each $i \leq n$, $j_{n,i}(\zeta_{\mathrm{L}}) = \zeta_{\mathrm{L}}$ and $j_{n,i}(\zeta_{\mathrm{H}}) = \zeta_{\mathrm{H}}$.*

We use the claim implicitly below, when talking about ζ_{L} where formally we should have its image by some embedding $j_{n,i}$.

Proof of Claim 7.7. Using the fact that $M_i \subset M$, and that any element of $M_i \| j_{n,i}(\kappa+\omega)$ is coded by an element of $M\|\kappa+\omega$, one can check that $\langle \zeta_{\mathrm{L}}, \zeta_{\mathrm{H}} \rangle$ is a pair of local indiscernibles for $j_{n,i}(\kappa)$ in M_i. $\langle j_{n,i}(\zeta_{\mathrm{L}}), j_{n,i}(\zeta_{\mathrm{H}}) \rangle$ is the lexicographically least such pair by the elementarity of $j_{n,i}$. From this and the fact that $j_{n,i}(\alpha) \geq \alpha$ for every ordinal α it follows that $\langle j_{n,i}(\zeta_{\mathrm{L}}), j_{n,i}(\zeta_{\mathrm{H}}) \rangle = \langle \zeta_{\mathrm{L}}, \zeta_{\mathrm{H}} \rangle$. ⊣

Lemma 7.8. *Suppose that x belongs to $M[h]$. Let $\vec{E}$ be an expanded rank progression wrt x. Suppose that* II *has a winning strategy in $G(x, \vec{E})$, and* I *has a winning strategy in $H(x, \vec{E})$.*

Let $n = \mathrm{lh}(\vec{E})$ and let l be the expectation at x. Suppose that $n < l$.

Then there exists a code $x^* \in M[h]$, and expanded rank progressions $\vec{E}^*$ and $\vec{F}$ wrt x and x^* respectively so that:

1. $\mathrm{lh}(\vec{E}^*) = n+1$, $\mathrm{lh}(P(\vec{E}^*)) = 0$, *and $\vec{E}^*$ extends $\vec{E}$;*
2. *x^* extends x and* $\mathrm{lh}(x^*) = \mathrm{lh}(x) + 1$*;*
3. *$\vec{F} = \vec{E}^* \| m$ where m meters x; and*
4. II *has a winning strategy in $G(x^*, \vec{F})$ and* I *has a winning strategy in $H(x^*, \vec{F})$.*

Proof. Let $\langle p_n, \dot{x}_n \rangle$ be the end-location of $r_n = r(\vec{E})$ if $n > 0$ (this is the same as the end location of s_n), or the empty location if $n = 0$. Note that x is then an n–extension of $\dot{x}_n[h]$.

Let $\dot{x}_{n+1} \in M$ be a $\mathrm{col}(\omega, {<}\kappa)$–name so that $\dot{x}_{n+1}[h] = x$. Fix $p_{n+1} \in h$ which forces a value for $\dot{x}_{n+1} \restriction n+1$, and forces $\dot{x}_{n+1}$ to be an n–extension of $\dot{x}_n$.

Let G denote the game $G(x, \vec{I}, \zeta_{\mathrm{L}})$ and let G' denote the game $G(x, \vec{I}, \zeta_{\mathrm{H}})$ (with both games computed in M_0). Since $n < l$, the games are played according to case 1 of Definition 4.5, illustrated in Diagram 5.

By assumption II has a winning strategy in G. Using the indiscernibility of ζ_{L} and ζ_{H} it follows that II has a winning strategy in G'. Let Σ be such a winning strategy.

Play $\zeta^* = \zeta_{\mathrm{L}}$ and the location $\langle p_{n+1}, \dot{x}_{n+1} \rangle$ fixed above as first moves for I in G'. The game continues subject to the rules of the basic rank game associated to κ, W_n, and $\langle p_{n+1}, \dot{x}_{n+1} \rangle$. Σ induces a strategy for I in this basic rank game. W_n is equal to $\mathrm{unrch}(r_n)$ by condition (5) of Definition 7.2, and r_n is according to Ψ by condition (2) of that definition. We are therefore in a position to apply Lemma 2.10. Let μ_{n+1}, U_{n+1}, and r_{n+1} be given by that lemma.

Let ν_{n+1}, V_{n+1}, and s_{n+1} be obtained similarly, working with the game $H = H(x, \vec{J}, \zeta_{\mathrm{L}})$ and a mirror image of Lemma 2.10.

By Remark 2.15, we have some freedom in the choice of μ_{n+1} and ν_{n+1}: any sufficiently strong measure would do in each case. We may therefore take $\mu_{n+1} = \nu_{n+1}$.

Let $\vec{E}^*$ be the expanded rank progression of length $n+1$ which extends $\vec{E}$ using the objects p_{n+1}, $\dot{x}_{n+1}$, μ_{n+1}, r_{n+1}, s_{n+1} fixed above, and the assignments $P_{n+1} = Q_{n+1} = \emptyset$. These assignments determine $\vec{E}^*$ completely.

Let $\vec{I}$ denote $\vec{I}(\vec{E}^*)$ and let $\vec{J}$ denote $\vec{J}(\vec{E}^*)$.

Our use of a winning strategy for II in G' guarantees that $(x, \vec{I}, \zeta_{\text{L}})$ is *not* good for I. This means that for all sufficiently large $\bar{\kappa} < \kappa$, player II has a winning strategy in $G^* = G^*(\bar{\kappa}, x, \vec{I}, \zeta_{\text{L}})$.

Similarly $(x, \vec{J}, \zeta_{\text{L}})$ is not good for II, and this means that for all sufficiently large $\bar{\kappa} < \kappa$, player I has a winning strategy in $H^* = H^*(\bar{\kappa}, x, \vec{J}, \zeta_{\text{L}})$.

Fix $\bar{\kappa} < \kappa$ large enough to witness both statements above, and large enough that x belongs to $M[h \restriction \bar{\kappa}]$. Let σ^* be a winning strategy for II in G^* and let τ^* be a winning strategy for I in H^*.

Let m meter x. Let $\bar{\delta}$ be the first Woodin cardinal of M above $\bar{\kappa}$. Let $\bar{g}$ be $\text{col}(\omega, \bar{\delta})$–generic/$M$ with $M[\bar{g}] = M[h \restriction \bar{\delta} + 1]$. Let $\dot{A}$ denote $\dot{A}(\bar{\delta}, \vec{I} \| m, \zeta_{\text{L}})$, and define $\dot{B}$ similarly. Let σ_{gen} and τ_{gen} be the generic and mirrored generic strategies maps associated to these names (using the generic $\bar{g}$).

σ^*, σ_{gen}, τ^*, and τ_{gen} combine (in the manner of the argument in [5, §1E]) to produce a real y in $M[h \restriction \bar{\delta} + 1]$ so that $x—, y$ belongs to neither $\dot{A}[\bar{g}]$ nor $\dot{B}[\bar{g}]$. Let $x^* = x—, y$ for this y. Finally, let $\vec{F} = \vec{E}^* \| m$.

One can now check that x^*, $\vec{E}^*$, and $\vec{F}$ satisfy the demands in the conclusion of Lemma 7.8. ⊣

Lemma 7.9. *Work under the assumptions of Lemma 7.8, but suppose that* $l \leq n$. *Let* g_l *be* $\text{col}(\omega, \delta(\vec{E} \restriction l))$*–generic over* $\text{Ult}(M, \mu(\vec{E} \restriction l))$*, and suppose that* $\vec{E} \restriction l$ *is a* g_l*-erp.*

Then there exists a code x^**, and expanded rank progressions* $\vec{E}^*$ *and* $\vec{F}$*, satisfying the conditions in the conclusion of Lemma 7.8, but with condition (1) of that lemma replaced with the following condition:*

1. (a) $\vec{E}^*$ *is a line extension of* $\vec{E} \restriction l$*,*
   (b) $\text{lh}(P(\vec{E}^*)) = \text{lh}(P(\vec{E} \restriction l)) + 1$*, and*
   (c) $\vec{E}^*$ *is a* g_l*-erp.*

Proof. Since $l \leq n$, the game $G(x, \vec{E})$ is played according to case 2 in Definition 4.5, and similarly with $H(x, \vec{E})$. The proof of the current lemma is similar to that of Lemma 7.8, only obtaining $\vec{E}^*$ through plays under case 2 (illustrated in Diagrams 6 and 11). These plays are constructed using a winning strategy for II in $G(x, \vec{E})$, a winning strategy for I in $H(x, \vec{E})$, and the generic and mirrored generic strategies associated to the names $\dot{A}(\vec{E} \restriction l)$ and $\dot{B}(\vec{E} \restriction l)$, where in both cases the generic used is g_l. We leave the precise details to the reader. ⊣

Using Lemmas 7.8 and 7.9 we will next construct an increasing sequence of codes and expanded rank progressions with respect to these codes. At the end of the construction we will be in a position to apply Claim 7.4, and through this application prove Lemma 3.3.

Definition 7.10. Let $x \in M[h]$ be a code and let $\vec{E}$ be an expanded rank progression of wrt x. We say that $\vec{E}$ is **suitable** for x if II has a winning strategy in $G(x, \vec{E})$ and I has a winning strategy in $H(x, \vec{E})$.

Definition 7.11. Let $\vec{E}$ be suitable for x. Let $n = \text{lh}(\vec{E})$. We say that $\vec{E}$ is **saturated** wrt x if there is *no* pair $\langle x^*, m \rangle$ so that:

1. $x^* \in M[h]$ extends x;
2. m meters $x^* \restriction \zeta$ for some $\zeta \in [\mathrm{lh}(x), \mathrm{lh}(x^*))$;
3. $m \leq n$ or $m < \max\{\mathrm{lh}(P_1(\vec{E})), \ldots, \mathrm{lh}(P_n(\vec{E}))\}$; and
4. $\vec{E}\|m$ is suitable for x^*.

DEFINITION 7.12. Let $\vec{E}$ be suitable for x. Let $n = \mathrm{lh}(\vec{E})$. Let $l \in (0, n]$. Let g_l be $\mathrm{col}(\omega, \delta(\vec{E}\restriction l))$–generic over $\mathrm{Ult}(M, \mu(\vec{E}\restriction l))$. Suppose that $\vec{E}\restriction l$ is a g_l–erp.

We say that $\vec{E}$ is g_l–**maximal** at l if there is *no* triple $\langle x^*, \vec{E}^*, \vec{F}\rangle$ so that:

1. $x^* \in M[h]$ extends x strictly;
2. l is expected at $x^* \restriction \xi$ for some $\xi \in [\mathrm{lh}(x), \mathrm{lh}(x^*))$;
3. $\vec{E}^*$ is a line extension of $\vec{E}\restriction l$;
4. $\vec{E}^*$ is a g_l–erp;
5. $\vec{F} = \vec{E}\|m$ where m is the least number which meters $x^* \restriction \xi$ for some $\xi \in [\mathrm{lh}(x), \mathrm{lh}(x^*))$; and
6. $\vec{F}$ is suitable for x^*.

We say that $\vec{E}$ is $\langle g_l \mid 0 < l \leq n\rangle$–maximal if it is g_l–maximal at l for each $l \in (0, n]$. $\vec{E}$ is **maximal** if it is $\langle g_l \mid 0 < l \leq n\rangle$ maximal for some sequence of generics $\langle g_l \mid 0 < l \leq n\rangle$.

CLAIM 7.13. *Suppose that $\vec{E}$ is suitable wrt x, and maximal. Let $n = \mathrm{lh}(\vec{E})$ and let l be the expectation at x. Then $l > n$.*

PROOF. Otherwise an application of Lemma 7.9 would contradict the maximality of $\vec{E}$ at l. ⊣

Claim 7.13 is crucial for our plans to find an increasing sequence of expanded rank progressions. It shows that given a maximal progression we can use Lemma 7.8, and this lemma gives an extension of the original progression. To continue this process inductively we must make the extension maximal too. This is done in the next lemma.

LEMMA 7.14. *Let $\vec{E}$ be suitable, saturated and maximal wrt a code $x \in M[h]$. Then there is a pair $\langle x^*, \vec{E}^*\rangle$ so that:*

1. *$x^* \in M[h]$ is a code which strictly extends x;*
2. *$\vec{E}^*$ is suitable, saturated and maximal wrt x^*; and*
3. *$\vec{E}^*$ strictly extends $\vec{E}$.*

PROOF. Let $n = \mathrm{lh}(\vec{E})$. If there is a code $x' \in M[h]$, extending x, so that x' is suitable for $\vec{E}$ and $n+1$ meters x', then fix such a code x'. Otherwise let $x' = x$. Notice that either way:

(∗) There is *no* code x^* so that x^* extends x' strictly, x^* is metered by $n+1$, and $\vec{E}$ is suitable for x^*.

Let l be the expectation at x'. By Claim 7.13, $l > n$. We may therefore apply Lemma 7.8. Let x'', $\vec{E}''$, and $\vec{F}$ be given by that lemma. Then $\vec{E}''$ extends $\vec{E}$ and $\vec{F} = \vec{E}''\|m$ where m meters x'. Note that $m > n$ and $m \geq \max\{\mathrm{lh}(P_1(\vec{E})), \ldots, \mathrm{lh}(P_n(\vec{E}))\}$, since otherwise we would have a contradiction to the saturation of $\vec{E}$. It follows that $\vec{F}$ extends $\vec{E}$.

Let $x_0 = x''$ and let $\vec{E}_0 = \vec{F}$. Let g be $\mathrm{col}(\omega, \delta(\vec{E}_0))$–generic over $\mathrm{Ult}(M, \mu(\vec{E}_0))$, with $M[g] \supset M[h]$.

Working inductively (in $M[g]$) construct $\vec{E}_k$ and x_k as follows:

- Extend x_k to a code $x'_k \in M[h]$ so that $x'_k \restriction \xi$ is metered by k for some $\xi \in [\mathrm{lh}(x_k), \mathrm{lh}(x'_k))$, and x'_k is suitable for $\vec{E}_k$ if possible. Otherwise set $x'_k = x_k$.
- Extend x'_k to a code $x''_k \in M[h]$ so that $x''_k \restriction \xi$ is metered by $n+1$ for some $\xi \in [\mathrm{lh}(x'_k), \mathrm{lh}(x''_k))$, and x''_k is suitable for $\vec{E}_k$ if possible. Otherwise set $x''_k = x'_k$.
- Extend x''_k to a code $x_{k+1} \in M[h]$ and line extend $\vec{E}_k$ to a g-erp $\vec{E}_{k+1}$, with $\mathrm{lh}(P(\vec{E}_{k+1})) = k+1$, $n+1$ expected at $x_{k+1} \restriction \xi$ for some $\xi \in [\mathrm{lh}(x''_k), \mathrm{lh}(x_{k+1}))$, and $\vec{E}_{k+1}$ suitable for x_{k+1}. If no such extension exists, then terminate the construction.

If the construction terminates, at stage k say, then $\vec{E}^* = \vec{E}_k$ and $x^* = x''_k$ witness the conclusion of Lemma 7.14, and we are done.

Suppose, towards a contradiction, that the construction does not terminate at any $k < \omega$. Let x_∞ code the union of the sequences coded by x_k, $k < \omega$. By construction $n+1$ is expected at $x_\infty \restriction \xi$ for cofinally many $\xi < \mathrm{lh}(x_\infty)$. So x_∞ is metered by $n+1$. By Claim 7.6, II has a winning strategy in $G(x_\infty, \vec{E}_0 \restriction n)$ and I has a winning strategy in $H(x_\infty, \vec{E}_0 \restriction n)$. In other words, $\vec{E}_0 \restriction n$ is suitable for x_∞, where this is interpreted over the model $M^* = \mathrm{Ult}(M, \mu(\vec{E}_0))$. But since $n+1$ meters x_∞, this contradicts condition $(*)$ above. ⊣

Let $\bar{x}_0$ code the empty sequence, and let $\vec{E}_0$ be the (unique) erp of length 0. $\vec{E}_0$ includes a value for W_0, and by Definition 7.2, this value is precisely $\mathrm{unrch}(\emptyset)$ (where "reachable" is relative to the strategy Ψ). Now Ψ is a strategy for II on $\bigcup_{W \text{ is good for I}} T_W$. By Claim 2.16, $T_{\mathrm{unrch}(\emptyset)}$ cannot belong to this union. So $W_0 = \mathrm{unrch}(\emptyset)$ is not good for I. In other words $(\bar{x}_0, \vec{I}(\vec{E}_0), \zeta_{\mathrm{L}})$ is not good for I.

By similar reasoning $(\bar{x}_0, \vec{J}(\vec{E}_0), \zeta_{\mathrm{L}})$ is not good for II.

Now an argument similar to the one ending the proof of Lemma 7.8 produces a real y in a small generic extension of M so that II has a winning strategy in $G(x_0, \vec{E})$ and I has a winning strategy in $H(x_0, \vec{E})$, where x_0 codes $\bar{x}_0$—$, y$. $\vec{E}_0$ is thus suitable for x_0. Since $\mathrm{lh}(\vec{E}_0) = 0$, $\vec{E}_0$ is trivially maximal and saturated. We are therefore in a position to apply Lemma 7.14. Let $\vec{E}_1$ and x_1 be given by that lemma. The lemma is specifically set up to allow iterated applications. Working inductively, let $\vec{E}_{k+1}$ and x_{k+1} be given by an application of Lemma 7.14 to $\vec{E}_k$ and x_k. We have, for each $k < \omega$:

1. $x_k \in M[h]$ is a code and $\vec{E}_k$ is suitable, saturated, and maximal wrt x_k;
2. $\vec{E}_{k+1}$ strictly extends $\vec{E}_k$.

Using Claim 7.4 we can now obtain infinite branches $\vec{d} \in [D]$ and $\vec{e} \in [E]$ so that $\vec{d}$ is according to Ψ, $\vec{e}$ is according to Ω, and $\rho(\vec{d}) = \chi(\vec{e})$. This completes the proof of Lemma 3.3

§8. Determinacy proofs. We say that a set $C \subset \mathbb{R}^{<\omega_1}$ belongs to a pointclass Γ just in case that the set (of reals) $\{x \mid x$ is a code for a sequence in $C\}$ belongs to Γ. In this section we use the lemmas of Section 3 to prove the determinacy of games $G_{\mathrm{adm}}(C)$ for C in Γ, for various pointclasses Γ.

THEOREM 8.1. *Suppose that there exists a class model M and a cardinal κ in M so that:*

(A) *κ is a limit of Woodin cardinals in M;*
(B) *For each $Z \subset M\|\kappa+1$ in M there is a measure $\mu \in M$ on κ so that $Z \in \mathrm{Ult}(M,\mu)$;*
(C) *$M\|\kappa+2$ is countable in* V*; and*
(D) *M is weakly iterable (see Neeman [5, Appendix A] for the definition).*

Then the games $G_{\mathrm{adm}}(C)$ are determined for all C in $<\omega^2-\Pi^1_1$.

Theorem 8.1 is optimal in the sense that the minimal inner model for assumptions (A)–(D) does not satisfy the statement "$G_{\mathrm{adm}}(C)$ is determined for all C in $<\omega^2-\Pi^1_1$."

PROOF OF THEOREM 8.1. Let D, E, ρ, and χ be the trees and embeddings of Section 3. Let $C \subset \mathbb{R}^{<\omega_1}$ be given, let $C^* \subset \mathbb{R}$ be the set of codes for elements of C, and suppose that C^* is $<\omega^2-\Pi^1_1$. We work to prove that $G_{\mathrm{adm}}(C)$ is determined.

The idea of the proof is quite simple. Consider the game where players I and II play on D to produce $\vec{d} \in [D]$, and in addition they play auxiliary moves following the rules in Martin's proof of determinacy for $<\omega^2-\Pi^1_1$ sets, with I trying to witness membership of $\rho(\vec{d})$ in C^*, and II trying to witness the opposite. Using Lemma 3.1 we will show that if I has a winning strategy in this game, then I wins $G_{\mathrm{adm}}(C)$. A similar argument using E and Lemma 3.2 will provide a condition under which II has a winning strategy in $G_{\mathrm{adm}}(C)$. Finally, a use of Lemma 3.3 will show that one of these conditions must hold.

Let us begin the argument. Let $u_0, u_1, \dots$ be the first ω uniform Silver indiscernibles for reals. Replacing M by $\mathrm{L}(M\|\kappa+2)$ we may assume that:

(E) M has the form $\mathrm{L}(M\|\kappa+2)$.

$u_0, u_1, \dots$ are then indiscernibles for M.

Let $\eta < \omega^2$ be such that C^* is $\eta-\Pi^1_1$. The case $\eta = 0$ is trivial. We may therefore assume $\eta > 0$. Increasing η if needed we may assume, for simplicity, that it is odd.

Let k be such that η belongs to the interval $(\omega\cdot k, \omega\cdot k+\omega]$. Let $\mathcal{C}$ be Martin's auxiliary games map for the set C^*, see Neeman [5, pp.74–75] for an outline of its definition. $\mathcal{C}$ assigns to each code x a game $\mathcal{C}[x]$, on ordinals below u_k, in a Lipschitz continuous manner.

Let $D^* \in M$ be the game in which I and II play on D to produce some $\vec{d} \in [D]$, and in addition to that play moves in $\mathcal{C}[\rho(\vec{d})]$. (Each round of D^* consists of a round in D followed by a move in $\mathcal{C}[\rho(\vec{D})]$. Note that in phrasing this format we are using the fact that both ρ and $\mathcal{C}$ are Lipschitz continuous.) Let $E^* \in M$ be the game in which I and II play on E to produce $\vec{e} \in [E]$, and in addition to that play moves in $\mathcal{C}[\chi(\vec{e})]$. In both games infinite runs are won by player I.

The games map $\mathcal{C}$ is definable in M from $u_0, \dots, u_k$. It follows that D^* and E^* are definable in M from the same parameters.

Claim 8.2. *If* I *has a winning strategy in D^* then (in* V*)* I *wins the game $G_{\mathrm{adm}}(C)$.*

Proof. Let $\Psi^* \in M$ be a winning strategy for I in D^*. We may take Ψ^* to be definable over M from $u_0, \dots, u_k$, since D^* is definable in this manner.

Say that a position $\langle \beta_0, \dots, \beta_{n-1} \rangle$ in $\mathcal{C}[x]$ is **good** (over M) if the ordinals played by II, namely the ordinals β_i for odd i, are indiscernibles for M, and the ordinals played by I are definable over M from the ordinals played by II and additional parameters in $\{u_0, \dots, u_k\} \cup M \| \kappa + 2$. This is an adaptation to the current settings of the definitions in [5, pp. 75,80].

Let Ψ, a strategy for I on D, be obtained from Ψ^* by ascribing indiscernible moves for II in the instances of $\mathcal{C}$ that come up during the play. Precisely, each position $\vec{d} \restriction n$ according to Ψ comes equipped with a sequence of ordinals $\langle \beta_0, \dots, \beta_{n-1} \rangle$ so that $\vec{d} \restriction n$ and $\vec{\beta} \restriction n$ together form a position according to Ψ^*, and so that $\vec{\beta} \restriction n$ is good. This requirement on positions according to Ψ determines the strategy Ψ uniquely.

Applying Lemma 3.1, let Σ be a winning strategy for I in $\widehat{G}_{\mathrm{adm}}(\Psi, M, \kappa)$. Let Γ be an iteration strategy for M. Σ and Γ together give rise to a strategy for I in $G_{\mathrm{adm}}(C)$. Let us denote this strategy by σ. We must check that σ is a winning strategy for I. Let $\vec{y}$ be a run of $G_{\mathrm{adm}}(C)$ played according to σ. We work to show that $\vec{y}$ belongs to C.

Let $\alpha = \mathrm{lh}(\vec{y})$ and let x code $\vec{y}$. As $\vec{y}$ is played according to σ, it comes with a corresponding run $\langle M^\xi, j^{\zeta,\xi} \mid \zeta \leq \xi < \alpha \rangle$ of $\widehat{G}_{\mathrm{adm}}(\Psi, M, \kappa)$. The run is played according to Σ, and the iteration maps $j^{\zeta,\xi}$ are all according to Γ.

Since Γ is an iteration strategy, each of the models M_ξ, $\xi \leq \alpha$, is wellfounded. Moreover, since the iteration is countable, the direct limit map $j^{1,\alpha}$ fixes the uniform indiscernibles.

Reading the payoff condition for $\widehat{G}_{\mathrm{adm}}(*)$ and using the fact that Σ is a winning strategy for I we see that there exists an infinite branch $\vec{d}$ through $j^{1,\alpha}(D)$ so that:

1. $\vec{d}$ is according to $j^{1,\alpha}(\Psi)$; and
2. $j^{1,\alpha}(\rho)(\vec{d}) = x$.

Now $j^{1,\alpha}(\Psi)$ is the strategy obtained from $j^{1,\alpha}(\Psi^*)$ by ascribing indiscernible moves for II. Using the properties of the game $j^{1,\alpha}(\mathcal{C})[j^{1,\alpha}(\rho)(\vec{d})]$ (see [5, pp. 75, 80], and [5, Fact 2D.11] for the specific properties we need; these properties are due to Martin [2]) it follows that $j^{1,\alpha}(\rho)(\vec{d})$ belongs to C^*. $j^{1,\alpha}(\rho)(\vec{d})$ is equal to x by condition (2). So $x \in C^*$, and hence $\vec{y} \in C$, as required. $\dashv$

Claim 8.3. *If* II *has a winning strategy in E^* then (in* V*)* II *wins the game $G_{\mathrm{adm}}(C)$.*

Proof. Similar to Claim 8.2, using this time Lemma 3.2, and revising the notion of a good position to reverse the roles of the players (so that I's moves are indiscernibles). The relevant facts on good positions now are the ones in [5, p.75]. The use of [5, Fact 2D.11] in the proof of Claim 8.2 is replaced here by a use of [5, Fact 2D.2], and this fact leads to the conclusion that $\vec{y}$ does *not* belong to C, and is therefore won by II, as required. ⊣

Claim 8.4. *It is impossible that* II *wins* D^* *and* I *wins* E^*.

Proof. Assume towards a contradiction that Ψ^* is a winning strategy for II in D^* and Ω^* is a winning strategy for I in E^*. Let Ψ, a strategy for II on D, be obtained from Ψ^* by ascribing indiscernible moves for I. Let Ω, a strategy for I on E, be obtained from Ω^* by ascribing indiscernible moves for II. Both Ψ and Ω are close to M. Applying Lemma 3.3, we find $\vec{d} \in [D]$ and $\vec{e} \in [E]$ so that

1. $\vec{d}$ is according to Ψ;
2. $\vec{e}$ is according to Ω; and
3. $\rho(\vec{d}) = \chi(\vec{e})$.

From condition (1) it follows (through an argument similar to the one in Claim 8.3) that $\rho(\vec{d}) \notin C^*$, and from condition (2) it follows (through an argument similar to the one ending the proof of Claim 8.2) that $\chi(\vec{e}) \in C^*$. This is a contradiction since $\rho(\vec{d}) = \chi(\vec{e})$ by condition (3). ⊣

D^* and E^* are both closed games, and hence determined. Claims 8.2, 8.3, and 8.4 together therefore imply that $G_{\text{adm}}(C)$ is determined. This completes the proof of Theorem 8.1. ⊣

A model M of a sufficiently large fragment of $\mathsf{ZFC}^*{-}\mathsf{Powerset}$ is said to satisfy adm(α) just in case that there is a cardinal κ in M so that, in M:

(A) κ is a limit of Woodin cardinals;
(B) For every $Z \subset M\|\kappa+1$ there is a measure μ on κ so that $Z \in \text{Ult}(M, \mu)$; and
(C) $M\|\kappa + \alpha = \mathcal{P}^\alpha(M\|\kappa)$ exists.

This terminology only makes sense for $\alpha \geq 2$, since we need $M\|\kappa{+}2$ for condition (B).

Theorem 8.5. *(For countable $\alpha \geq 1$.) Suppose that there exists a weakly iterable model M which satisfies* adm$(1 + \alpha + 1)$*, with $M\|\kappa + 2$ countable in* V*. Then the games $G_{\text{adm}}(C)$ are determined for all $C \subset \mathbb{R}^{<\omega_1}$ in $\Delta^0_{\alpha+4}$.*

Proof. Let $C \subset \mathbb{R}^{<\omega_1}$ be given, let C^* be the set of codes for positions in C, and suppose that C^* is $\Delta^0_{\alpha+4}$. We work to show that $G_{\text{adm}}(C)$ is determined.

Let D, E, ρ, and χ be the trees and embeddings of Section 3. Note D and E are trees on $M\|\kappa{+}2$. Observe that $A = \rho^{-1}(C^*) \subset [D]$ and $B = \chi^{-1}(C^*) \subset [E]$ are $\Delta^0_{\alpha+4}$. Let $\{A_l\}_{l<\omega}$ be an enumeration of all Π^0_α subsets of $[D]$ which are needed to define A. Let $\{B_l\}_{l<\omega}$ be an enumeration of all the Π^0_α subsets of $[E]$ needed to define B. Working in M Let D^* be a cover (in the sense of Martin [3]) of D which unravels all sets in $\{A_k\}_{l<\omega}$. M contains exactly enough levels of the Von Neumann hierarchy above $M\|\kappa + 2$ to unravel these sets, and D^* is a tree on $M\|\kappa + 1 + \alpha + 1$. Similarly let E^* be a cover of E which unravels $\{B_l\}_{l<\omega}$. Let $A^* \subset [D^*]$ be the pre-image of A, and let $B^* \subset [E^*]$ be the pre-image of B. Let $G_{D^*}(A^*)$ denote the game on D^* with payoff A^*, and define $G_{E^*}(B^*)$ similarly. A^* is a Δ^0_4 subset of $[D^*]$ and B^* is a Δ^0_4 subset of $[E^*]$. It follows that in M, both $G_{D^*}(A^*)$ and $G_{E^*}(B^*)$ are determined. The following three claims are thus enough to establish the determinacy of $G_{\text{adm}}(C)$.

CLAIM 8.6. *If (in M) I wins $G_{D^*}(A^*)$ then (in V) I wins $G_{\mathrm{adm}}(C)$.*

CLAIM 8.7. *If (in M) II wins $G_{E^*}(B^*)$ then (in V) II wins $G_{\mathrm{adm}}(C)$.*

CLAIM 8.8. *It is impossible that (in M) II wins $G_{D^*}(A^*)$ and I wins $G_{E^*}(B^*)$.*

The proofs of these claims are similar to the proofs of Claims 8.2, 8.3, and 8.4 respectively. The only difference is that now we convert from Ψ^* (respectively Ω^*) to Ψ (respectively Ω) not using indiscernibles, but rather using the fact that D^* is a cover of D (respectively, E^* is a cover of E). ⊣ (Theorem 8.5.)

The next theorem improves Theorem 8.5 in the case that $\alpha < \omega$, reducing the assumption $\mathrm{adm}(1+\alpha+1)$ to $\mathrm{adm}(\alpha+1)$.

THEOREM 8.9. *(For countable $\alpha \geq 1$.) Suppose that there exists a weakly iterable model M which satisfies $\mathrm{adm}(\alpha+1)$, with $M\|\kappa+2$ countable in V. Then the games $G_{\mathrm{adm}}(C)$ are determined for all $C \subset \mathbb{R}^{<\omega_1}$ in $\Delta^0_{\alpha+4}$.*

PROOF. Let D, E, ρ, and χ be the trees and embeddings of Section 3. It is enough to show that Π^0_1 subsets of D (respectively E) can be unraveled by a cover using a tree on $M\|\kappa+2$. With this one can modify the proof of Theorem 8.5 so that D^* and E^* are trees on $M\|\kappa+\alpha+1$, instead of $M\|\kappa+1+\alpha+1$, allowing for a proof that uses $\mathrm{adm}(\alpha+1)$ instead of $\mathrm{adm}(1+\alpha+1)$.

D and E are trees on $M\|\kappa+2$, and in general one cannot expect to unravel subsets of trees on $M\|\kappa+2$ using trees still on $M\|\kappa+2$. But the trees D and E have a special property that makes this possible. Specifically, for the tree D say:

1. Player II's moves come from $M\|\kappa+1$;
2. $\rho(\vec{d})$ depends only on player II's moves in $\vec{d}$;
3. The future of the game from a position in which II had just played depends only on II's moves in that position.

Using these properties of D, one can construct a *demi-cover* D^* of D which unravels a given countable collection of Π^0_1 sets, and which uses a tree on $M\|\kappa+2$. A demi-cover is very much like a cover, except that in lifting a play $\vec{d} \in [D]$ to a play $\vec{d}^* \in [D^*]$ one is allowed to *change* I's moves in $\vec{d}$. Such a change is harmless by conditions (2) and (3). It is condition (1) that allows keeping the demi-cover a tree on $M\|\kappa+2$. We omit the actual construction, and refer the reader to Neeman [4, 8.3–8.7] for more information on demi-covers. ⊣

Remark 8.10. The argument of Theorems 8.5 and 8.9 holds for payoff sets in the relativized pointclass $\Delta^0_{\alpha+4}(z)$ for $z \in M$. Since any real z can be absorbed into a small generic extension of an iterate of M, the conclusion of the theorem can be strengthened to apply to the boldface pointclass, $\boldsymbol{\Delta}^0_{\alpha+4}$. A similar remark holds for Theorem 8.1.

COROLLARY 8.11. *Let M satisfy $\mathrm{adm}(\alpha+1)$. Then the statement "$G_{\mathrm{adm}}(C)$ is determined for every C in $\boldsymbol{\Delta}^0_{\alpha+4}$" holds in M.*

PROOF. This follows from the proofs of Theorems 8.5 and 8.9. Note that $G_{D^*}(A^*)$ and $G_{E^*}(B^*)$ are determined inside M. Working inside M, find a countable elementary submodel H which has the winning strategies in these games, satisfies enough of ZFC$^*-$Powerset for the results of Section 3, and satisfies $\mathrm{adm}(\alpha+1)$. Let $\bar{M}$ be the transitive collapse of H. Inside M, the model $\bar{M}$ is weakly iterable. Working inside M apply Claims 8.6, 8.7, and 8.8 to $\bar{M}$, and conclude that $G_{\mathrm{adm}}(C)$ is determined. $\dashv$

Corollary 8.11 is optimal in the sense that $\boldsymbol{\Delta}^0_{\alpha+4}$ cannot be replaced by $\boldsymbol{\Sigma}^0_{\alpha+4}$. The minimal iterable inner model for $\mathrm{adm}(\alpha+1)$ is not a model of the statement "$G_{\mathrm{adm}}(C)$ is determined for all C in $\boldsymbol{\Sigma}^0_{\alpha+4}$." This can be seen through a computation of the complexity of the canonical wellordering of the reals in this inner model. The key point in the computation is that the relevant comparison process can be presented as a game ending at the first admissible relative to the play, using a combination of the methods of Steel [8, Proof of 1.10] and Steel [7].

Corollary 8.11 holds for $\alpha \geq 1$, as $\mathrm{adm}(\beta)$ only makes sense for $\beta \geq 2$. It therefore covers the pointclasses $\boldsymbol{\Delta}^0_\gamma$ for $\gamma \geq 5$. For $\gamma \leq 4$, optimal proofs of the determinacy of $G_{\mathrm{adm}}(C)$ for all C in $\boldsymbol{\Delta}^0_\gamma$ are not known.

Let M and κ satisfy assumptions (A)–(C) in Section 3. Let D, E, ρ, and χ be the trees and embeddings of Section 3.

Let $\mathbb{A} \in M$ be a forcing notion, and let a be $\mathbb{A}$–generic/M. Let $\dot{\Psi} \in M$ be an $\mathbb{A}$–name and suppose that $\dot{\Psi}[a]$ is a strategy for I on D.

Define $\widehat{G}'_{\mathrm{adm}}(\dot{\Psi}, M, \kappa)$ to be played according to the rules of $\widehat{G}_{\mathrm{adm}}$ described in Section 3, but with the following modified payoff: I wins just in case that there exist $\vec{d}$ and a^α so that:

0. a^α is $\mathbb{A}^\alpha$–generic/M^α where $\mathbb{A}^\alpha = j^{1,\alpha}(\mathbb{A})$;
1. $\vec{d}$ is according to Ψ^α where $\Psi^\alpha = j^{1,\alpha}(\dot{\Psi})[a^\alpha]$; and
2. $j^{1,\alpha}(\rho)(\vec{d}) = x$.

$j^{1,\alpha}\colon M \to M^\alpha$ is the iteration embedding created through the moves in $\widehat{G}'_{\mathrm{adm}}$, see Section 3. The difference between the payoff here and the payoff in Section 3 is in the addition of a^α, and the fact that in condition (1) we do not take $j^{1,\alpha}(\Psi)$, where $\Psi = \dot{\Psi}[a]$, but instead shift $\dot{\Psi}$ by $j^{1,\alpha}$ and interpret it using a generic over M^α.

Lemma 8.12. *Suppose that $\dot{\Psi}[a]$ is a strategy for player* I *on D. Suppose that both $M\|\kappa+2$ and $\mathcal{P}^M(\mathbb{A})$ are countable in* V. *Then (in* V*)* I *wins the game $\widehat{G}'_{\mathrm{adm}}(\dot{\Psi}, M, \kappa)$.*

Proof. This is similar to the proof in Section 5, with the following modifications:

Perps should be redefined to include objects $\{\bar{a}_i\}_{i\leq n}$, with each a_i an element of M_i.

Definition 5.4 should be revised to add the following condition on expanded rank progressions: $\bar{a}_i$ is a condition in $j_{n,i}(\mathbb{A})$. Further, the clause "played according to $j_{n,i}(\Psi)$" in condition (2) of Definition 5.4 should be revised to say that "$\bar{a}_i$ forces that r_i is played according to $j_{n,i}(\dot{\Psi})$." Finally, a clause should be added to condition (4) in Definition 5.4 requiring that $\bar{a}_{i+1}$ extends $j^{-1}_{i+1,i}(\bar{a}_i)$.

Claim 5.6 should be revised to say that r is according to $\dot{\Psi}[a]$ where $a = \bigcup_{n<\omega}(\bar{a}(\vec{E}_n))$.

And last, but not least, a new book-keeping element should be added to the construction, for choosing the objects $\bar{a}_i$ at each stage of the construction. This should be done in such a way that the object $a = \bigcup_{n<\omega}(\bar{a}(\vec{E}_n))$ generated by the application of Claim 5.6 at the end of the argument gives rise to a generic filter over M^α. $\dashv$

Suppose now that $\delta > \kappa$, and $M \models$"δ is a Woodin cardinal." Let $\dot{A} \in M$ be a $\mathrm{col}(\omega,\delta)$–name for a set of reals. Define the game $\widehat{G}(\mathrm{adm}+1, \dot{A}, M, \kappa, \delta)$ to be played as follows: The game starts by following the rules of $\widehat{G}_{\mathrm{adm}}$, to produce $\vec{y} = \langle y_\xi \mid \xi < \alpha\rangle$, and an iterate M^α of M with iteration embedding $j^{1,\alpha}\colon M \to M^\alpha$. (Note that $\mathrm{L}_\alpha[y_\xi \mid \xi < \alpha]$ is then admissible.) The game continues with precisely one additional mega-round. In this one extra mega-round player I plays a length ω iteration tree $\mathcal{T}^\alpha$ on M^α, and player II plays a cofinal branch b^α through $\mathcal{T}^\alpha$. This ends the game. We set $M^{\alpha+1}$ equal to the direct limit along b^α, set $j^{\alpha,\alpha+1}$ equal to the direct limit embedding, and define $j^{1,\alpha+1}$ by composition. Player I wins just in case that there exists some g so that:

1. g is $\mathrm{col}(\omega, j^{1,\alpha+1}(\delta))$–generic/$M^{\alpha+1}$; and
2. $x \in j^{1,\alpha+1}(\dot{A})[g]$, where x codes $\vec{y}$.

Given a $\mathrm{col}(\omega,\delta)$–name $\dot{B} \in M$ for a set of reals we define the mirror image game $\widehat{H}(\mathrm{adm}+1, \dot{B}, M, \kappa, \delta)$. This game starts by following $\widehat{H}_{\mathrm{adm}}$, and continues with II playing $\mathcal{T}^\alpha$ and I playing b^α. (Both must satisfy the demands listed above, in the definition of $\widehat{G}(\mathrm{adm}+1, \dots)$.) We define $M^{\alpha+1}$ and $j^{1,\alpha+1}$ as above. Player II wins this run of $\widehat{H}(\mathrm{adm}+1, \dots)$ just in case that there exists some h so that:

1. h is $\mathrm{col}(\omega, j^{1,\alpha+1}(\delta))$–generic/$M^{\alpha+1}$; and
2. $x \in j^{1,\alpha+1}(\dot{B})[h]$, where x codes $\vec{y}$.

THEOREM 8.13. *Assume that M and κ satisfy conditions (A)–(C) in Section 3, and that $\delta > \kappa$ is a Woodin cardinal of M. Suppose that $\mathrm{V}^M_{\delta+2}$ is countable in* V *and let g be $\mathrm{col}(\omega,\delta)$–generic/$M$. Let $\dot{A} \in M$ and $\dot{B} \in M$ be $\mathrm{col}(\omega,\delta)$–names for sets of reals. Then at least one of the three cases below holds:*

1. *(In* V*)* I *wins the game $\widehat{G}(\mathrm{adm}+1, \dot{A}, M, \kappa, \delta)$;*
2. *(In* V*)* II *wins the game $\widehat{H}(\mathrm{adm}+1, \dot{B}, M, \kappa, \delta)$; or*
3. *In $M[g]$, there exists some code x so that x belongs to neither $\dot{A}[g]$ nor $\dot{B}[g]$.*

Moreover, M can distinguish which of these conditions holds. Precisely, there are formulae $\phi_{\mathrm{I}}(\mathrm{adm}+1,*)$ and $\phi_{\mathrm{II}}(\mathrm{adm}+1,*)$ (defined uniformly over all M, κ, and δ) so that: If $M \models \phi_{\mathrm{I}}(\mathrm{adm}+1, \dot{A}, \kappa, \delta)$ then condition (1) hold; if $M \models \phi_{\mathrm{II}}(\mathrm{adm}+1, \dot{B}, \kappa, \delta)$ then condition (2) holds; and otherwise condition (3) holds.

PROOF. Let D, E, ρ, and χ be the trees and projections of Section 3. Let $\mathcal{A}$ be the auxiliary games map (see Neeman [5, §1A]) associated to $(M\|\delta) \times \dot{A}$, δ, and $X = M\|\kappa+2$. Let $\mathcal{B}$ be the mirrored auxiliary games map (see [5, §1D]) associated to $(M\|\delta) \times \dot{B}$, δ, and $X = M\|\kappa+2$.

Define D^* to be the game in which players I and II play on D to produce $\vec{d} \in [D]$, and in addition to that play moves in $\mathcal{A}[\rho(\vec{d})]$. More precisely, round n

of D^* starts with moves in round n of D, and continues with moves in round n of $\mathcal{A}[\rho(\vec{d}\restriction n+1)]$. (In phrasing this format we are using the Lipschitz continuity of ρ and the Lipschitz continuity of the auxiliary games map $\mathcal{A}$.) Infinite runs of D^* are won by player II.

Define E^* similarly, with the tree E, the projection χ, and the mirrored auxiliary games map $\mathcal{B}$. Infinite runs of E^* are won by player I.

Lemma 8.14. *If (in M) I wins D^*, then (in V) I wins $\widehat{G}(\mathrm{adm}+1, \dot{A}, \kappa, \delta)$.*

Proof. Let $\mathbb{A} = \mathrm{col}(\omega, M\|\delta+1)$, and let a be $\mathbb{A}$–generic over M. Let $\Psi^* \in M$ be a winning strategy for I in D^*.

Claim 8.15. *There is a strategy Ψ on D in $M[a]$, so that: for every play $\vec{d}$ according to Ψ, there exists a length ω iteration tree $\mathcal{T}$ on M, so that for every wellfounded cofinal branch b through $\mathcal{T}$, there is $\mathrm{col}(\omega, j_b(\delta))$–generic g over M_b with $\rho(\vec{d}) \in j_b(\dot{A})[g]$.*

Proof. This is a direct application of the methods of Neeman [5, §1E]. Let $\varphi \in M[a]$ enumerate $M\|\delta+1$ in order type ω. Let $\mathcal{A}_{\mathrm{piv}}$ be the pivot strategies map (see [5, §1C]) corresponding to $\mathcal{A}$, and let σ_{piv} be the corresponding pivot strategies map.

Define Ψ by using σ_{piv} to ascribe moves for II in (shifts of) $\mathcal{A}_{\mathrm{piv}}$. Precisely, each run $\vec{d}$ of Ψ comes equipped with a run $\mathfrak{P} = (\mathcal{T}, \vec{a})$ of $\mathcal{A}_{\mathrm{piv}}[\varphi, \rho(\vec{d})]$, played according to $\sigma_{\mathrm{piv}}[\varphi, \rho(\vec{d})]$, and so that for each $n < \omega$, $\vec{d}\restriction n$ and $\vec{a}\restriction n$ form a position of the shift of D^* to the $2n$th model of $\mathcal{T}$, according to the shift of Ψ^* to that model. This property of runs of Ψ determines Ψ completely.

It is easy to check, using the methods in [5, Chapter 1], that the strategy Ψ described above satisfies the requirement of the claim. $\dashv$

Let $\dot{\Psi}$ be the canonical $\mathbb{A}$–name for the strategy Ψ given by the last claim. Lemma 8.14 now follows through an application of Lemma 8.12 with the name $\dot{\Psi}$, followed by an application of the property of Ψ given by Claim 8.15 (or more precisely the shift of this claim to M^α). $\dashv$

Lemma 8.16. *If (in M) II wins E^*, then (in V) II wins $\widehat{H}(\mathrm{adm}+1, \dot{B}, \kappa, \delta)$.*

Proof. This is a precise mirror image of the previous lemma. $\dashv$

Lemma 8.17. *If (in M) II wins D^* and I wins E^*, then there exists a code $x \in M[g]$ so that x belongs to neither $\dot{A}[g]$ nor $\dot{B}[g]$.*

Proof. We work throughout the proof in $M[g]$. Let Ψ^* be a strategy for II in D^* and let Ω^* be a strategy for I in E^*. Let σ_{gen} be the generic strategies map corresponding to $\mathcal{A}$, and let τ_{gen} be the mirrored generic strategies map corresponding to $\mathcal{B}$, see Neeman [5, §§1B,1D]. σ_{gen} and τ_{gen} belong to $M[g]$.

Ψ^* and σ_{gen} combine to produce a strategy Ψ for player II in D, with the property that every run $\vec{d}$ according to Ψ comes equipped with a run $\vec{a}$ of $\mathcal{A}[\rho(\vec{d})]$, according to $\sigma_{\mathrm{gen}}[\rho(\vec{d})]$, so that $\vec{d}$ and $\vec{a}$ together form a run of D^* according to Ψ^*.

Ω^* and τ_{gen} similarly combine to produce a strategy Ω for I in E.

Applying Lemma 3.3 we get $\vec{d} \in [D]$ and $\vec{e} \in [E]$ so that $\vec{d}$ is according to Ψ, $\vec{e}$ is according to Ω, and $\rho(\vec{d}) = \chi(\vec{e})$. By absoluteness we may find such $\vec{d}$ and $\vec{e}$ in $M[g]$. Let $x = \rho(\vec{d}) = \chi(\vec{e})$.

From the fact that $\vec{d}$ is according to Ψ it follows that there exists an infinite run $\vec{a}$ of $\mathcal{A}[\rho(\vec{d})]$ according to $\sigma_{\text{gen}}[\rho(\vec{d})]$. $\vec{a}$ is a generic run of $\mathcal{A}[\rho(\vec{d})]$. Using [5, Lemma 1B.2] it follows that $\rho(\vec{d}) \notin \dot{A}[g]$.

A similar argument shows that $\chi(\vec{e}) \notin \dot{B}[g]$.

The code $x = \rho(\vec{d}) = \chi(\vec{e})$ therefore witnesses the truth of Lemma 8.17. $\dashv$

D^* and E^* are both determined in M, being open and closed respectively. Lemmas 8.14, 8.16, and 8.17 therefore demonstrate that at least one of the conditions (1), (2), (3) in Theorem 8.13 must hold. The formulae ϕ_{I} and ϕ_{II} can be obtained by noticing that the games D^* and E^* above are definable in M from κ, δ, $\dot{A}$ (for D^*), and $\dot{B}$ (for E^*). $\dashv$ (Theorem 8.13.)

Let $C \subset \mathbb{R}^{<\omega_1}$. Define $G(\text{adm} + \theta, C)$ to be played as follows: players I and II collaborate in the usual fashion to produce reals y_ξ, until reaching the first α so that $\mathrm{L}_\alpha[y_\xi \mid \xi < \alpha]$ is admissible. They then continue to play, in the usual fashion, to produce reals $y_{\alpha+\xi}$ for $\xi < \theta$. The game ends with a sequence $\langle y_\xi \mid \xi < \alpha + \theta\rangle$, and player I wins iff this sequence belongs to C.

By the **code** for a sequence $\langle y_\xi \mid \xi < \alpha + \theta\rangle$ produced through a play of $G(\text{adm} + \theta, C)$ we mean the sequence $\langle x\rangle^\frown\langle y_\xi \mid \xi \in [\alpha, \theta)\rangle$ where x codes $\langle y_\xi \mid \xi < \alpha\rangle$ (in the sense of Section 1). $C \subset \mathbb{R}^{<\omega_1}$ is Γ **in the codes** if the set of codes for elements of C belongs to Γ.

THEOREM 8.18. *Let θ be a countable ordinal. Suppose that there exists a weakly iterable class model M and $\kappa \in M$ so that:*

(A) *κ is a limit of Woodin cardinals in M;*
(B) *For every $B \subset M\|\kappa + 1$ in M there exists a measure $\mu \in M$ on κ so that $Z \in \mathrm{Ult}(M, \mu)$;*
(C) *There are (in order) θ Woodin cardinals $\langle \delta_\xi \mid \xi < \theta\rangle$ of M above κ; and*
(D) *$M\| \sup_{\xi<\theta}(\delta_\xi + 1)$ is countable in* V.

Then the games $G(\text{adm} + \theta, C)$ are determined for all C in $<\omega^2 - \Pi^1_1$.

PROOF. Combine Theorem 8.13 with the results of [5, Chapter 2]. $\dashv$

Theorem 8.18 too is optimal, in the sense that the determinacy proved does not hold inside the minimal iterable class model for conditions (A)–(C) of the theorem.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA LOS ANGELES
LOS ANGELES, CA 90095-1555
E-mail: ineeman@math.ucla.edu