Sol'n's to sample final problems (not checked carefully)

Note Title 5/31/2016

1. $pt = (1,2,3)$

direction vector $= (1,0,2)\times(0,1,3) = \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = (-2,-3,1)$

$\vec{r}(t) = (1,2,3) + t(-2,-3,1)$

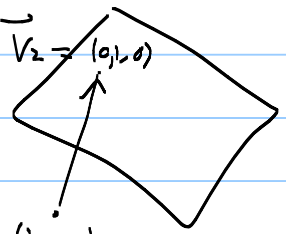
2. Compute volume of parallelepiped and check whether vol $= 0$.

$$\text{vol} = \begin{pmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{pmatrix} = 1(18) - 4(36) - 7(-18) = 0.$$ Yes.

3. $$\cos\theta = \frac{(1,1,0)\cdot(1,1,1)}{|(1,1,0)|\cdot|(1,1,1)|} = \frac{2}{\sqrt{2}\sqrt{3}} = \frac{2}{\sqrt{6}} \qquad \theta = \cos^{-1}\frac{2}{\sqrt{6}}.$$

4. Pick pt $(0,1,0)$ on $5x+y-z=1$.

$(\frac{1}{2},0,0)$ on $10x+2y-2z=5$ $\quad \vec{v}_2 = (0,1,0)$



$\vec{v}_1 = (\frac{1}{2},0,0)$

Component / proj of $\vec{v}_2 - \vec{v}_1$ to $\vec{n} = \frac{(5,1,-1)}{\sqrt{27}}$ unit normal

$$\left|(-\tfrac{1}{2},1,0)\cdot\vec{n}\right| = \left|(-\tfrac{1}{2},1,0)\cdot\frac{(5,1,-1)}{\sqrt{27}}\right| = \left|\frac{-\frac{5}{2}+1}{\sqrt{27}}\right| = \frac{3}{2\cdot 3\sqrt{3}} = \frac{1}{2\sqrt{3}}.$$

5. $$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$ $\quad \frac{\text{degree numerator} = 2}{\text{degree denominator} = 2}$

Test $x=0$: $\displaystyle\lim_{y\to 0}\frac{0}{y^2} = 0$

$x=y$: $\displaystyle\lim_{x\to 0}\frac{x^2}{x^2+x^2} = \lim_{x\to 0}\frac{x^2}{2x^2} = \lim_{x\to 0}\frac{1}{2} = \frac{1}{2}$

$\Big\}$ different, so limit DNE

$\lim_{(x,y)\to 0} \frac{xy}{\sqrt{x^2+y^2}}$ $\frac{\text{deg num} = 2}{\text{deg denom} = 1}$

Use squeeze thm $0 \le \frac{|xy|}{\sqrt{x^2+y^2}} \le \frac{|xy|}{|x|} = |y|$

lim $\downarrow$ 0 $\downarrow$ lim 0

$\therefore \lim_{(x,y)\to 0} \frac{xy}{\sqrt{x^2+y^2}} = 0.$

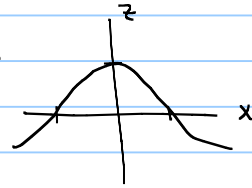
$\lim_{(x,y)\to 0} \frac{x^2-y^2}{x+y}$ $\frac{\text{deg num} = 2}{\text{deg denom} = 1}$

" $\lim_{(x,y)\to 0} \frac{(x-y)(x+y)}{x+y} = \lim_{(x,y)\to 0} (x-y) = 0.$

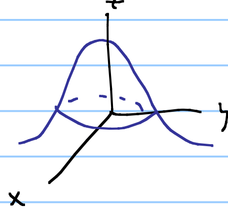
6. Surface $\underbrace{x^2+y^2}_{r}+z^3=1.$

Surface of revolution : Sketch in xz-plane and rotate along z-axis

$x^2+z^3=1$



$\longrightarrow$



Tgt plane at $(0,-3,2)$. $f(x,y,z) = x^2+y^2+z^3$, $\nabla f = (2x, 2y, 3z^2)$

$\vec{n} = \nabla f(0,-3,2) = (0,-6,12)$

$(0,-6,12)(x-0, y+3, z-2) = 0$

$-6(y+3) + 12(z-2) = 0$

7. (a) $\vec{r}'(t) = (6\sqrt{2}t, 1-9t^2, 2\sqrt{2})$, $|\vec{r}'(t)| = \sqrt{72t^2+(1-9t^2)^2+8} = \sqrt{81t^4+54t^2+9}$
$= \sqrt{9(3t^2+1)^2} = 3(3t^2+1)$

length $= \int_0^1 |\vec{r}'(t)|\,dt = \int_0^1 3(3t^2+1)\,dt = 3(t^3+t)\Big|_0^1 = 3$

(b) Solve for s,t in: $(3\sqrt{2}t^2, t-3t^3, 2\sqrt{2}t) = (3s, s^2-4, s^3)$.

(A) $3\sqrt{2}t^2 = 3s \implies s = \sqrt{2}t^2$ ①

(B) $t - 3t^3 = s^2 - 4$

(C) $2\sqrt{2}t = s^3 \overset{①}{\implies} 2\sqrt{2}t = 2\sqrt{2}t^6 \implies t^6 - t = 0 \implies t(t^5-1) = 0 \implies t=0$ or $t=1$

When $t=0$, $s=0$ by ①. Does not satisfy (B), so not a soln.

When $t=1$, $s=\sqrt{2}$ by ①. (B) is satisfied: $1-3(1)^3 \overset{?}{=} \sqrt{2}^2 - 4$. Hence $t=1$, $s=\sqrt{2}$ is a soln.

Paths intersect at $(3\sqrt{2}, -2, 2\sqrt{2})$, but particles do not collide since $t \neq s$.

8. ① $\vec{r}(t) = (\cos t, t, \sin t)$

$\vec{r}'(t) = (-\sin t, 1, \cos t)$ $|\vec{r}'(t)| = \sqrt{2}$

$s(t) = \int_0^{x=t} \sqrt{2}\,dx = \sqrt{2}x\Big|_0^t = \sqrt{2}t$ $\therefore$ $t = s/\sqrt{2}$ s = arclength parameter

$\vec{r}(s) = (\cos\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}})$ ✓

② $\vec{r}'(s) = (-\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}})$, $\vec{r}''(s) = (-\frac{1}{2}\cos\frac{s}{\sqrt{2}}, 0, \frac{1}{2}\sin\frac{s}{\sqrt{2}})$

$K(s) = |\vec{r}''(s)| = \frac{1}{2}$

$x = s+t, \quad y = s-t$

9. $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\cdot 1 + \frac{\partial f}{\partial y}\cdot 1, \quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot 1 + \frac{\partial f}{\partial y}\cdot(-1)$

$\Rightarrow \frac{\partial f}{\partial s}\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2$

$\frac{1}{2}\left(\left(\frac{\partial f}{\partial s}\right)^2 + \left(\frac{\partial f}{\partial t}\right)^2\right) = \frac{1}{2}\left((f_x+f_y)^2 + (f_x-f_y)^2\right)$

$= \frac{1}{2}\cdot 2 f_x^2 + f_y^2 = |\nabla f|^2.$

10. (a) $f(x,y,z) = \sqrt{x+y^2+z^2} \qquad (x,y,z) = (11,5,8) \qquad (\Delta x, \Delta y, \Delta z) = (-.01, -.02, .01)$

$\Delta f = \frac{1}{2}(x+y^2+z^2)^{-1/2}(\Delta x + 2y\,\Delta y + 2z\,\Delta z) \qquad \sqrt{11+5^2+8^2} = 10$

$= \frac{1}{2}\frac{1}{10}(-.01 + -.2 + .16)$

$= \frac{1}{20}(-.05) = -.0025$

$\therefore\ 9.9975$

(b) $V(r,h) = \frac{\pi}{3}r^2h \qquad (r,h) = (5,10) \qquad (\Delta r, \Delta h) = (2,-1)$

$\Delta V \approx \frac{\pi}{3}(2rh\,\Delta r + r^2\Delta h)$

$= \frac{\pi}{3}(100(2) + 25(-1)) = \frac{\pi}{3}(175)$

11. $\nabla f = (3y, 3x, 2z) \qquad \vec{u} = \frac{(-1,-2,-3)}{\sqrt{14}}$

$D_{\vec{u}} f(1,2,3) = \vec{u}\cdot\nabla f(1,2,3)$

$= \frac{(-1,-2,-3)}{\sqrt{14}}\cdot(6,3,6)$

$= \frac{-6-6-18}{\sqrt{14}} = -\frac{30}{\sqrt{14}}$

12. $f(x,y) = 100 - .4x^2 - .3y^2$

Direction of steepest descent $= -\nabla f(1,1) = -(-.8x, -.6y)\big|_{(x,y)=(1,1)} = (.8, .6).$

$\nabla f = (-.8x, -.6y)$

13. $y = y(x) \qquad y(x)^3 + 3x^4 = 0$

$3y^2\frac{dy}{dx} + 12x^3 = 0$

$\frac{dy}{dx} = \frac{-12x^3}{3y^2} = -4\frac{x^3}{y^2}$

14. $f(x,y) = y^2x - yx^2 + xy$

(a) $f_x = y^2 - 2xy + y = 0$ $\qquad$ $f_y = 2xy - x^2 + x = 0$ $\qquad$ $D = \begin{vmatrix} -2y & 2y-2x+1 \\ 2y-2x+1 & 2x \end{vmatrix}$

$y(y-2x+1) = 0$ $\qquad$ $x(2y-x+1) = 0$

$\Rightarrow y=0$ or $y = 2x-1$ $\qquad$ $x=0$ or $y = \frac{1}{2}(x-1)$ $\qquad$ $= -4xy - (2y-2x+1)^2$

① $y=0 \Rightarrow -x^2+x=0 \Rightarrow x = 0$ or 1 $\qquad$ $(0,0), (1,0)$

② $x=0 \Rightarrow y^2+y=0 \Rightarrow y = 0$ or -1 $\qquad$ $(0,0), (0,-1)$

③ $y = 2x-1 = \frac{1}{2}(x-1)$

$\frac{3}{2}x = \frac{1}{2}$ $\quad$ $x = \frac{1}{3}$, $\quad y = -\frac{1}{3}$ $\qquad$ $(\frac{1}{3}, -\frac{1}{3})$

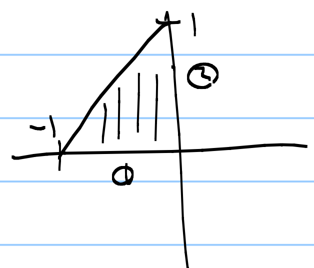
$(0,0)$ $\quad D<0$

$(1,0)$ $\quad D<0$ $\quad$ saddles

$(0,-1)$ $\quad D<0$

$(\frac{1}{3}, -\frac{1}{3})$ $\quad D = -4(\frac{1}{3})(-\frac{1}{3}) - (-\frac{1}{3})^2$

$= \frac{3}{4} > 0$ $\qquad f_{xx} > 0$ $\qquad$ min.

(b) 

Check bdry

Crit. pts. in D: $(0,0), (0,-1)$.

$f=0$ here

① $y=0$ $\quad -1 \le x \le 0$ $\qquad f=0$

② $x=0$ $\qquad f=0$ $\qquad$ max

③ $y = x+1$ $\quad g(x) = (x+1)^2 \cdot x - (x+1)x^2 + x(x+1)$

$= (\cancel{x^3}+2x^2+x) - (\cancel{x^3}+\cancel{x^2}) + (\cancel{x^2}+x)$

$= 2x^2+2x = 0$

$g'(x) = 4x+2 = 0 \Rightarrow x = -\frac{1}{2}$ $\qquad y = \frac{1}{2}$

crit. pt $(-\frac{1}{2}, \frac{1}{2})$ $\qquad f(-\frac{1}{2}, \frac{1}{2}) = 2(-\frac{1}{2})^2 + 2(-\frac{1}{2}) = \frac{1}{2} - 1 = -\frac{1}{2}.$ $\qquad$ min.

15. $f(x,y) = (x^2+1)y$, $g(x,y) = x^2+y^2=5$

$\nabla f = \lambda \nabla g$

$(2xy,\ x^2+1) = \lambda(2x, 2y)$

Solve $\begin{cases} 2xy = 2x\lambda \Rightarrow \overset{①}{x=0} \text{ or } \overset{②}{\lambda = y} \\ x^2+1 = 2y\lambda \\ x^2+y^2=5 \end{cases}$

① $x=0 \Rightarrow y = \pm\sqrt{5}$ $\qquad (0, \pm\sqrt{5})$

② $\lambda = y$ $\quad \begin{matrix} x^2+1=2y^2 \\ x^2+y^2=5 \end{matrix} \Rightarrow x^2 = 2y^2-1 \Rightarrow (2y^2-1)+y^2=5$

$3y^2 = 6 \quad y = \pm\sqrt{2}$

$x = \pm\sqrt{3}.$ $\qquad \pm(\sqrt{3}, \sqrt{2}), \pm(-\sqrt{3}, \sqrt{2}).$

$(0, \sqrt{5})$ $\quad f = \sqrt{5}$

$(0, -\sqrt{5})$ $\quad f = -\sqrt{5}$

$(\pm\sqrt{3}, \sqrt{2})$ $\quad f = 4\sqrt{2} \approx 5.6$

$(\pm\sqrt{3}, -\sqrt{2})$ $\quad f \approx -5.6$

max & min

16. P passes thru $(1,1,1)$ means $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1.$

Maximize $f(a,b,c) = abc$ (ignore the $1/6$)

Constraint $g(a,b,c) = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1.$

$\nabla f = \lambda \nabla g$

$(bc, ac, ab) = \lambda\left(-\frac{1}{a^2}, -\frac{1}{b^2}, -\frac{1}{c^2}\right).$

$bc = -\frac{1}{a^2}\lambda \Rightarrow \lambda = -a^2bc$

$ac = -\frac{1}{b^2}\lambda \Rightarrow \lambda = -ab^2c$

$\Rightarrow a = b.$ Similarly $b = c.$

$ab = -\frac{1}{c^2}\lambda \Rightarrow \lambda = -abc^2$

Since $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$, $a=b=c=3$

$\therefore$ Smallest value of V is $\frac{1}{6}(3)(3)(3) = \frac{9}{2}$

17. $\nabla f = \lambda \nabla g$ $\qquad g(x,y,z) = x^2+y^2+z^2 = 36$

$(0, 2y, -10) = \lambda(2x, 2y, 2z)$

① $0 = 2x\lambda \Rightarrow \cancel{\lambda = 0}$ or $x = 0$ $\qquad \lambda = 0$ contradicts ③

② $2y = 2y\lambda \Rightarrow y(\lambda - 1) = 0 \Rightarrow y = 0$ or $\lambda = 1$

③ $-10 = 2z\lambda$ $\qquad$ ④ $\qquad$ ③

④ $x^2+y^2+z^2 = 36$ $\qquad z = \pm 6 \qquad z = -5$

$\therefore$ Candidates $(0,0,6), (0,0,-6), (0,\sqrt{11},-5), (0,-\sqrt{11},-5)$ $\qquad$ ④ $\quad y = \pm\sqrt{11}$

f values $\quad -60 \quad 60 \quad 61 \quad 61$

min value $\qquad$ max values

18. $f(x,y,z)=3x^2+y$, $\quad g(x,y,z)=4x-3y=9$ ④, $\quad h(x,y,z)=x^2+z^2=9$ ⑤

$\nabla f=\lambda\nabla g+\mu\nabla h$

$(6x,1,0)=\lambda(4,-3,0)+\mu(2x,0,2z)$.

① $6x=4\lambda+2x\mu$

② $1=-3\lambda \Rightarrow \lambda=-\frac{1}{3}$

③ $0=2z\mu \Rightarrow \mu=0$ or $z=0$.

If $\lambda=-\frac{1}{3}$, $\mu=0$, then ① $\Rightarrow 6x=-\frac{4}{3} \Rightarrow x=-\frac{2}{9} \overset{④}{\Rightarrow} 4\left(-\frac{2}{9}\right)-3y=9$

$\Rightarrow -9-\frac{8}{9}=3y \Rightarrow y=\frac{89}{27}$. ⑤ $\Rightarrow z^2=9-\frac{4}{81} \Rightarrow z=\pm\sqrt{9-\frac{4}{81}}$

$f\left(-\frac{2}{9},\frac{89}{27},\pm\sqrt{9-\frac{4}{81}}\right)=\frac{12}{81}+\frac{89}{27}$ min.

If $\lambda=-\frac{1}{3}$, $z=0$, ⑤ $\Rightarrow x=\pm 3$

$x=+3 \overset{④}{\Rightarrow} 12-3y=9 \Rightarrow y=1$

$x=-3 \Rightarrow -12-3y=9 \Rightarrow y=-7$

$f(3,1,0)=27+1=28$ max

$f(-3,-7,0)=27-7=20$

19. Lagrange multipliers

$f(x,y,z)=x^2+y^2+z^2$, $\quad g(x,y)=x^2+y^2=1$ ④, $\quad h(x,y)=x+z=1$ ⑤

$(2x,2y,2z)=\lambda(2x,2y,0)+\mu(1,0,1)$

① $2x=2x\lambda+\mu$

② $2y=2y\lambda \Rightarrow y=0$ or $\lambda=1$

③ $2z=\mu$

If $y=0$, then ④ $\Rightarrow x=\pm 1$ $\xrightarrow{x=1} z=0$, $\xrightarrow{x=-1} z=2$

If $\lambda=1$, ① $\Rightarrow \mu=0 \overset{③}{\Rightarrow} z=0 \overset{⑤}{\Rightarrow} x=1 \overset{④}{\Rightarrow} y=0$

Candidates: $(1,0,0)$, $(-1,0,2)$

$f=1$, 5

min, max

Using parametrization $\quad x=\cos\theta$, $y=\sin\theta$, $z=1-x=1-\cos\theta$

$\text{Distance}^2=f(x,y,z)=x^2+y^2+z^2=1+(1-\cos\theta)^2=2-2\cos\theta+\cos^2\theta$

$f'(\theta)=2\sin\theta-2\cos\theta\sin\theta=0$

$\Rightarrow \sin\theta(1-\cos\theta)=0 \Rightarrow \sin\theta=0$ or $\cos\theta=1$

$\theta=0,\pi$ $\qquad \theta=0$

$\theta=0 \Rightarrow (1,0,0)$ closest

$\theta=\pi \Rightarrow (-1,0,2)$ farthest.