Sample problems for midterm exam

(1) Let $\mathbf{v} = (1, 2, 3)$ and $\mathbf{w} = (1, 1, 0)$.
   (a) Find the unit vector in the same direction as $\mathbf{v}$.
   (b) Find the angle between the vectors $\mathbf{v}$ and $\mathbf{w}$.
   (c) Let $\mathcal{L}_1$ be the line through $(0, 0, 0)$ in the direction of $\mathbf{v}$ and $\mathcal{L}_2$ be the line through $(-1, 0, 3)$ in the direction of $\mathbf{w}$. Write down the parametric equations of $\mathcal{L}_1$ and $\mathcal{L}_2$. Then determine whether these lines intersect.

(2) Find an equation of the plane through the points $(1, 1, 1)$, $(0, 1, 1)$, and $(-1, -1, -1)$.

(3) Find the volume of the parallelepiped spanned by $\mathbf{u} = (2, 2, 1)$, $\mathbf{v} = (1, 0, 3)$, and $\mathbf{w} = (0, -4, 0)$.

(4) Given the vectors $\mathbf{a} = (4, -1, 5)$ and $\mathbf{b} = (2, 1, 1)$, find the decomposition $\mathbf{a} = \mathbf{a}_{\|\mathbf{b}} + \mathbf{a}_{\perp\mathbf{b}}$.

(5) Prove that the vectors $(1, 4, 7)$, $(2, 5, 8)$, and $(3, 6, 9)$ are parallel to the same plane. Find an equation of such a plane.

(6) Let $\mathcal{L}$ be the intersection of the planes $x - y - z = 1$ and $2x + 3y + z = 2$. Find parametric equations for the line $\mathcal{L}$.

(7) Find the distance from the point $Q = (1, 1, 1)$ to the plane $2x + y + 5z = 2$.

(8) Let C be a curve parametrized by $x(t) = \sin t$, $y(t) = 3\cos t$. Sketch this curve. Find all points on the curve at which the tangent line to the curve is parallel to the line $y = x$.

(9) Find the location at $t = 3$ of a particle whose path satisfies $\mathbf{r}'(t) = (2t - \frac{1}{(t+1)^2}, 2t - 4)$ and $\mathbf{r}(0) = (3, 8)$.

(10) The curve $\mathbf{r}(t) = (e^t \cos 4t, e^t \sin 4t)$ has the property that the angle ψ between the position vector and the tangent vector is constant. Find the angle ψ.

(11) Using the same curve $\mathbf{r}(t) = (e^t \cos 4t, e^t \sin 4t)$, evaluate $s(t) = \displaystyle\int_{-\infty}^{t} |\mathbf{r}'(u)|du$. (It is convenient to take the lower limit $-\infty$ because $\mathbf{r}(-\infty) = (0, 0)$.) Then find an arc length parametrization of $\mathbf{r}(t)$.

(12) Compute the curvature function of the parametrization $\mathbf{r}(t) = (a\cos t, b\sin t)$ of the ellipse $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$.

(13) Find the curvature of $\mathbf{r}(t) = (\cosh t, \sinh t, t)$ at $t = 0$.

(14) A ball is thrown from the ground at the angle of $\frac{\pi}{6}$ with the ground at the initial velocity of 10 m/s. The gravitational acceleration is 9.8 m/s^2. Find how long it takes before the ball hits the ground again.

(15) Sketch the surfaces in Exercises 25–38 from Section 13.6.

(16) The temperature in 3-space is given by $T(x, y, z) = x^2 - y^2 - z$. Draw isotherms corresponding to the temperatures $T = -1, 0, 1$.