Sample problems for midterm exam

(1) Let $\mathbf{v} = (1, 2, 3)$ and $\mathbf{w} = (1, 1, 0)$.

(a) Find the unit vector in the same direction as $\mathbf{v}$.

Solution: $\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(1,2,3)}{\sqrt{1^2+2^2+3^2}} = (\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}})$.

(b) Find the angle between the vectors $\mathbf{v}$ and $\mathbf{w}$.

Solution: The angle θ satisfies $\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}|\cdot|\mathbf{w}|} = \frac{3}{\sqrt{14}\sqrt{2}}$, so $\theta = \cos^{-1}(\frac{3}{\sqrt{28}})$.

(c) Let $\mathcal{L}_1$ be the line through $(0, 0, 0)$ in the direction of $\mathbf{v}$ and $\mathcal{L}_2$ be the line through $(-1, 0, 3)$ in the direction of $\mathbf{w}$. Write down the parametric equations of $\mathcal{L}_1$ and $\mathcal{L}_2$. Then determine whether these lines intersect.

Solution: The lines are given by $\mathbf{r}_1(s) = (0, 0, 0)+s(1, 2, 3)$ and $\mathbf{r}_2(t) = (-1, 0, 3)+t(1, 1, 0)$. We set $\mathbf{r}_1(s) = \mathbf{r}_2(t)$ and solve for s, t. This gives $s = -1 + t$, $2s = t$, $3s = 3$. This has a solution $s = 1$, $t = 2$. Hence the lines intersect at $(1, 2, 3)$.

(2) Find an equation of the plane through the points $(1, 1, 1)$, $(0, 1, 1)$, and $(-1, -1, -1)$.

Solution: A normal vector $\mathbf{n}$ is given by $\mathbf{n} = ((1, 1, 1) - (0, 1, 1)) \times ((1, 1, 1) - (-1, -1, -1)) = (1, 0, 0) \times (2, 2, 2) = (0, -2, 2)$. Choose a point, say $(1, 1, 1)$, on the plane. Then the equation is $(0, -2, 2)\cdot((x, y, z)-(1, 1, 1)) = 0$, or $-2(y-1)+2(z-1) = 0$ or $-2y + 2z = 0$.

(3) Find the volume of the parallelepiped spanned by $\mathbf{u} = (2, 2, 1)$, $\mathbf{v} = (1, 0, 3)$, and $\mathbf{w} = (0, -4, 0)$.

Solution: The volume is given by the determinant $\begin{vmatrix} 2 & 2 & 1 \\ 1 & 0 & 3 \\ 0 & -4 & 0 \end{vmatrix} = 20$.

(4) Given the vectors $\mathbf{a} = (4, -1, 5)$ and $\mathbf{b} = (2, 1, 1)$, find the decomposition $\mathbf{a} = \mathbf{a}_{\parallel\mathbf{b}} + \mathbf{a}_{\perp\mathbf{b}}$.

Solution: $\mathbf{a}_{\parallel\mathbf{b}} = (\mathbf{a} \cdot \frac{\mathbf{b}}{|\mathbf{b}|})\frac{\mathbf{b}}{|\mathbf{b}|} = ((4, -1, 5) \cdot \frac{1}{\sqrt{6}}(2, 1, 1))\frac{1}{\sqrt{6}}(2, 1, 1) = (4, 2, 2)$. $\mathbf{a}_{\perp\mathbf{b}} = \mathbf{a} - \mathbf{a}_{\parallel\mathbf{b}} = (4, -1, 5) - (4, 2, 2) = (0, -3, 3)$.

(5) Prove that the vectors $(1, 4, 7)$, $(2, 5, 8)$, and $(3, 6, 9)$ are parallel to the same plane. Find an equation of such a plane.

Solution: You can either compute the volume spanned by the three vectors $\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = 0$ or observe that $(2,5,8)-(1,4,7)=(1,1,1)$ and $(3,6,9)-(1,4,7)=(2,2,2)$, so the three vectors, viewed as points in 3-space, lie on the same line.

To compute the equation of the plane, compute $(1,4,7)\times(1,1,1)=(-3,6,-3)$. After rescaling we use $\mathbf{n}=(1,-2,1)$. We then get $(x-1)-2(y-4)+(z-7)=0$.

(6) Let $\mathcal{L}$ be the intersection of the planes $x-y-z=1$ and $2x+3y+z=2$. Find parametric equations for the line $\mathcal{L}$.

Solution: Eliminate one variable for example by adding the two equations: $3x+2y=1$. Then $y=\frac{1}{2}-\frac{3}{2}x$. Plugging back into $x-y-z=1$, we have $z=x-y-1=-\frac{3}{2}+\frac{5}{2}x$. Using x as t, we get $\mathbf{r}(t)=(t,\frac{1}{2}-\frac{3}{2}t,-\frac{3}{2}+\frac{5}{2}t)$.

(7) Find the distance from the point $Q=(1,1,1)$ to the plane $2x+y+5z=2$.

Solution: Pick a point on the plane, say $P=(1,0,0)$. Then compute the (magnitude of the) projection of $Q-P=(0,1,1)$ to the unit normal $\mathbf{n}=\frac{(2,1,5)}{\sqrt{30}}$ as follows: $(0,1,1)\cdot\frac{(2,1,5)}{\sqrt{30}}=\frac{6}{\sqrt{30}}$.

(8) Let C be a curve parametrized by $x(t)=\sin t$, $y(t)=3\cos t$. Sketch this curve. Find all points on the curve at which the tangent line to the curve is parallel to the line $y=x$.

Solution: This is an ellipse centered at $(0,0)$ with major axis of length 3 along the y-axis and minor axis of length 1 along the x-axis.

Take $\mathbf{r}'(t)=(\cos t,-3\sin t)$ and solve for t in $\cos t=-3\sin t$. Then $\tan t=-\frac{1}{3}$. Hence we either have $\cos t=\frac{3}{\sqrt{10}}$, $\sin t=-\frac{1}{\sqrt{10}}$ or $\cos t=-\frac{3}{\sqrt{10}}$, $\sin t=\frac{1}{\sqrt{10}}$. The desired points are $\pm(-\frac{1}{\sqrt{10}},\frac{9}{\sqrt{10}})$.

(9) Find the location at $t=3$ of a particle whose path satisfies $\mathbf{r}'(t)=(2t-\frac{1}{(t+1)^2},2t-4)$ and $\mathbf{r}(0)=(3,8)$.

Solution: Taking antiderivatives, we get $\mathbf{r}(t)=(t^2+(t+1)^{-1},t^2-4t)+\mathbf{A}$. Plugging in $\mathbf{r}(0)=(3,8)$, we obtain $\mathbf{A}=(2,8)$. Finally, $\mathbf{r}(3)=(11\frac{1}{4},5)$.

(10) The curve $\mathbf{r}(t)=(e^t\cos 4t,e^t\sin 4t)$ has the property that the angle ψ between the position vector and the tangent vector is constant. Find the angle ψ.

Solution: Taking derivatives, $\mathbf{r}'(t)=(e^t(\cos 4t-4\sin 4t),e^t(\sin 4t+4\cos 4t))$. The angle ψ satisfies $\cos\psi=\frac{\mathbf{r}(t)\cdot\mathbf{r}'(t)}{|\mathbf{r(t)}|\cdot|\mathbf{r}'(\mathbf{t})|}=\frac{e^{2t}}{e^t\cdot\sqrt{17}e^t}=\frac{1}{\sqrt{17}}$. Hence $\psi=\cos^{-1}(\frac{1}{\sqrt{17}})$.

(11) Using the same curve $\mathbf{r}(t) = (e^t \cos 4t, e^t \sin 4t)$, evaluate $s(t) = \displaystyle\int_{-\infty}^{t} |\mathbf{r}'(u)|du$. (It is convenient to take the lower limit $-\infty$ because $\mathbf{r}(-\infty) = (0, 0)$.) Then find an arc length parametrization of $\mathbf{r}(t)$.

Solution: From the previous exercise, $|\mathbf{r}'(u)| = \sqrt{17}e^u$. Then $s(t) = \sqrt{17}e^u|_{u=-\infty}^{u=t} = \sqrt{17}e^t$. Solving for t in terms of s, $t = \ln \frac{s}{\sqrt{17}}$. Then plug back in to $\mathbf{r}(t)$.

(12) Compute the curvature function of the parametrization $\mathbf{r}(t) = (a \cos t, b \sin t)$ of the ellipse $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$.

Solution: We use the formula $\kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$. $\mathbf{r}(t) = (a \cos t, b \sin t, 0)$, $\mathbf{r}'(t) = (-a \sin t, b \cos t, 0)$, $\mathbf{r}''(t) = (-a \cos t, -b \sin t, 0)$. Then $|\mathbf{r}'(t) \times \mathbf{r}''(t)| = ab$, $|\mathbf{r}'(t)| = (a^2 \sin^2 t + b^2 \cos^2 t)^{1/2}$, and $\kappa(t) = \frac{ab}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}}$.

(13) Find the curvature of $\mathbf{r}(t) = (\cosh t, \sinh t, t)$ at $t = 0$.

Solution: We will use $\frac{d}{dt}(\cosh t) = \sinh t$, $\frac{d}{dt}(\sinh t) = \cosh t$, and $\cosh^2 t - \sinh^2 t = 1$. $\mathbf{r}'(t) = (\sinh t, \cosh t, 1)$. $\mathbf{r}''(t) = (\cosh t, \sinh t, 0)$. $|\mathbf{r}'(t)\times\mathbf{r}''(t)| = \sqrt{1 + \sinh^2 t + \cosh^2 t}$. $|\mathbf{r}'(t)| = \sqrt{1 + \sinh^2 t + \cosh^2 t}$. Hence $\kappa(t) = \frac{1}{1+\sinh^2 t+\cosh^2 t}$ and $\kappa(0) = \frac{1}{2}$.

(14) A ball is thrown from the ground at the angle of $\frac{\pi}{6}$ with the ground at the initial velocity of 10 m/s. The gravitational acceleration is 9.8 m/s^2. Find how long it takes before the ball hits the ground again.

Solution: $\mathbf{a}(t) = (0, -9.8)$, so $\mathbf{v}(s) = (0, -9.8t)+\mathbf{v}(0)$, where $\mathbf{v}(0) = (10 \cos \frac{\pi}{6}, 10 \sin \frac{\pi}{6}) = (5\sqrt{3}, 5)$. Next, $\mathbf{r}(t) = (5\sqrt{3}t, -4.9t^2 + 5t)$. (We are assuming that $\mathbf{r}(0) = (0, 0)$.) Solving for $-4.9t^2 + 5t = 0$, we obtain $t = 0$ or $t = \frac{5}{4.9}$, which is the desired solution.

(15) Sketch the surfaces in Exercises 25–38 from Section 13.6.

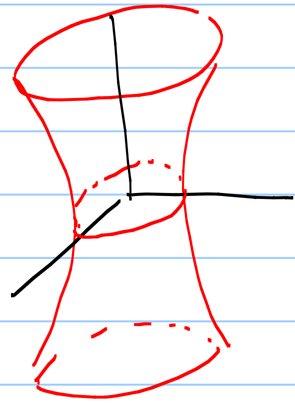
See figures on the next pages.

(16) The temperature in 3-space is given by $T(x, y, z) = x^2 - y^2 - z$. Draw isotherms corresponding to the temperatures $T = -1, 0, 1$.

See figures on the next pages.

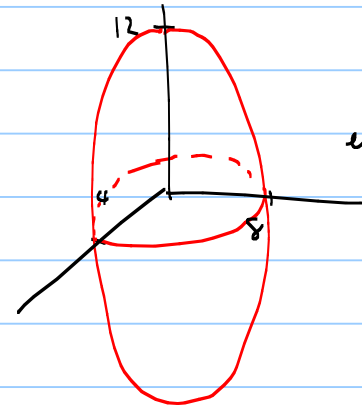
Note Title 5/1/2017

13.6 #25



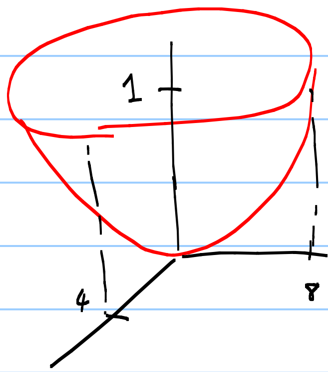
hyperboloid of one sheet

#26



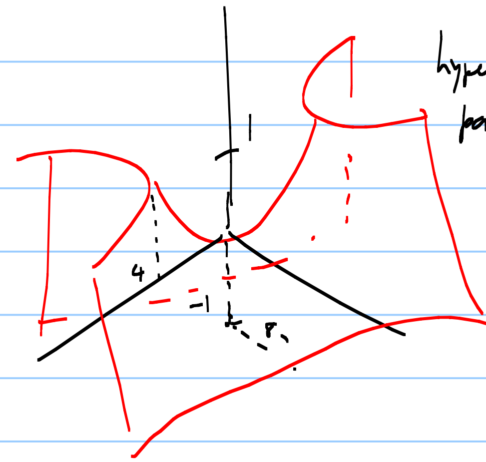
ellipsoid

#27



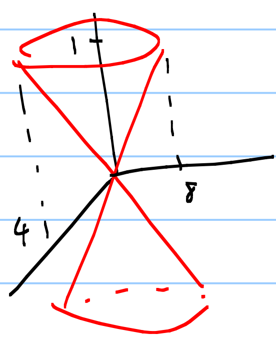
paraboloid

#28



hyperbolic paraboloid

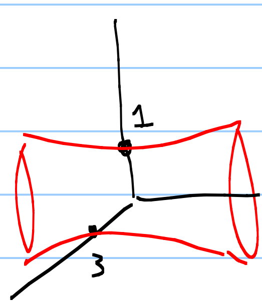
#29



cone

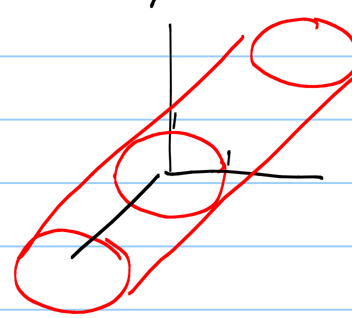
#30 cylinder over a parabola

#31

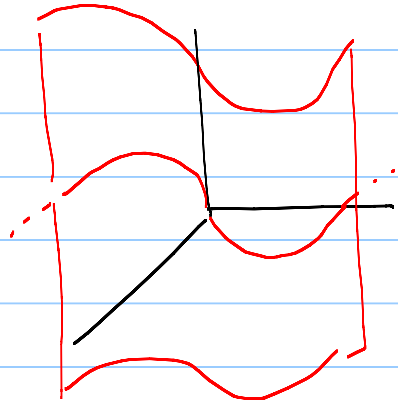


hyperboloid of 1 sheet

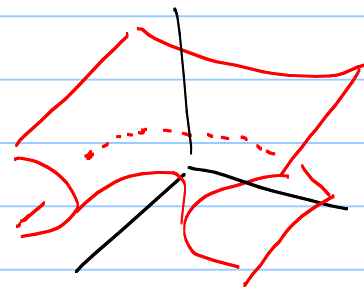
#32 Circular cylinder



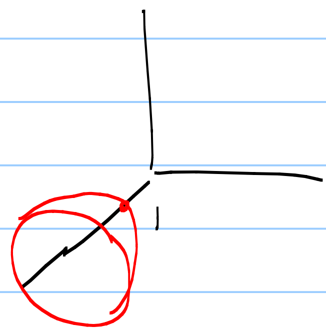
#33 cylinder over sine curve



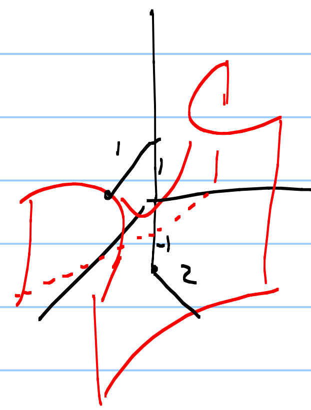
#34 hyperbolic paraboloid



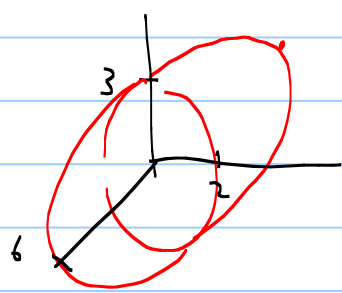
#35 paraboloid



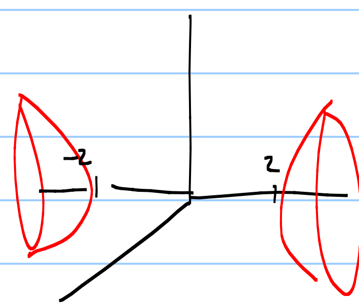
#36 hyperbolic paraboloid



#37 ellipsoid



#38 hyperboloid of 2 sheets



#16 $T=-1$ $\quad -1 = x^2-y^2-z$ $\qquad T=0,\ 0 = x^2-y^2-z$ $\qquad T=1,\ 1 = x^2-y^2-z$

$T=-1$ and $T=1$ are just $z=\mp 1$ translates of $z=x^2-y^2$