Sample problems for final exam

(1) Determine a parametrization of the line L with the following properties:
   (a) L passes through the point $(1, 2, 3)$.
   (b) L is orthogonal to $(1, 0, 2)$ and to $(0, 1, 3)$.
(2) Do the vectors $(1, 4, -7)$, $(2, -1, 4)$, and $(0, -9, 18)$ lie on the same plane?
(3) Find the angle between $(1, 1, 0)$ and $(1, 1, 1)$.
(4) Find the distance between the parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.
(5) Find the limit if it exists, or show that the limit does not exist.
   (a) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$.
   (b) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$.
   (c) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y}$.
(6) Sketch the surface $x^2 + y^2 + z^3 = 1$. Then find the equation of the tangent plane to this surface at $(0, 3, -2)$.
(7) A particle travels along a path $\mathbf{r}(t) = (3\sqrt{2}t^2, t - 3t^3, 2\sqrt{2}t)$.
   (a) What is the length of the path traced by the particle between time $t = 0$ and $t = 1$?
   (b) A second particle travels along the path $\mathbf{r}_1(t) = (3t, t^2 - 4, t^3)$. Find all the points at which the paths of the two particles intersect, and, for each such point, determine whether the particles collide at that point.
(8) Find the arc length parametrization of $\mathbf{r}(t) = (\cos t, t, \sin t)$. Use it to compute the curvature.
(9) Let $x = s + t$ and $y = s - t$. For any differentiable function $f(x, y)$, verify that the following relationships are true:
   (a) $(\frac{\partial f}{\partial x})^2 - (\frac{\partial f}{\partial y})^2 = \frac{\partial f}{\partial s}\frac{\partial f}{\partial t}$.
   (b) $|\nabla f|^2 = \frac{1}{2}((\frac{\partial f}{\partial s})^2 + (\frac{\partial f}{\partial t})^2)$.
(10) Estimate the following:
   (a) $\sqrt{10.99 + 4.98^2 + 8.01^2}$.
   (b) The change in volume of a right circular cone of radius 5 and height 10 that results from increasing the radius by 2 and decreasing the height by 1. (Recall that $V = \frac{\pi}{3}r^2h$.)
(11) Find the directional derivative of $f(x, y, z) = 3xy + z^2$ at the point $(1, 2, 3)$ in the direction of the origin from that point. (Use the unit vector in the direction of the origin.)
(12) Suppose you are hiking on a hill which locally has elevation given by $100 - .4x^2 - .3y^2$ and you are at the point with (x, y)-coordinates $(1, 1)$. What direction do you head in order to go down the hill in the steepest direction?
(13) Let y be a function of x. Find $\frac{dy}{dx}$ in terms of x, y if $y^3 + 3x^4 = 0$.
(14) Let $f(x, y) = y^2x - yx^2 + xy$.

(a) Find all the critical points of f and determine whether each is a local minimum, local maximum, or saddle point.

(b) Find the global maximum and minimum of f on the domain $D = \{(x, y) \mid -1 \leq x \leq 0, 0 \leq y \leq x + 1\}$.

(15) What is the maximum value that $f(x, y) = (x^2 + 1)y$ takes on the circle $x^2 + y^2 = 5$?

(16) A plane P with equation $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, $a, b, c > 0$, forms a tetrahedron with the coordinate planes of volume $V = \frac{1}{6}abc$. Assuming that P passes through $(1, 1, 1)$, find the smallest possible value of V.

(17) Find the maximum and minimum values of $f(x, y, z) = y^2 - 10z$, subject to the constraint $x^2 + y^2 + z^2 = 36$.

(18) Find the maximum and minimum values of $f(x, y, z) = 3x^2 + y$, subject to the constraints $4x - 3y = 9$ and $x^2 + z^2 = 9$.

(19) Parametrize the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + z = 1$. Find the point on this intersection which is farthest from the origin (do in two ways: using the parametrization, or using Lagrange multipliers).