from deRham to Morse

Tianyu Yuan 2019/05

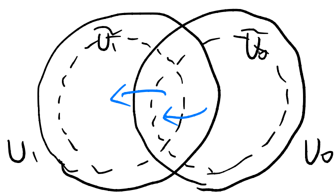
We will prove the quasi-equivalence between dg-category $\mathrm{Open}(X)$ and A_∞-category $\mathrm{Fuk}(T^*X)$ generated by $\{\Gamma_{U,df}\}$

Def Morse(X)

ob: (U, m), where m is defining function for U

$\hom((U_0,m_0),(U_1,m_1)) = \mathbb{K}\cdot \mathrm{Cr}(\tilde{U}_0 \cap \tilde{U}_1,\ f_1 - \varepsilon_0 f_0)$

where $\tilde{U}_i \subset U_i$, ε_0 small s.t. $\nabla(f_1 - \varepsilon_0 f_0)$ enters from $\partial\tilde{U}_1$ and leaves from $\partial\tilde{U}_0$, $f = \log m$



Def. Fuk(T^*X),

ob: $\{\Gamma_{U,df}\}$

$\hom(\Gamma_{U_0,df_0}, \Gamma_{U_1,df_1}) = \mathbb{K}\cdot\{\tilde{\Gamma}_0 \cap \tilde{\Gamma}_1\}$

where $\tilde{\Gamma}_i$ is perturbation of Γ_i

Then by Fukaya-Oh's theorem & Nadler Zaslow, we see:

Thm F: Morse$(X) \xrightarrow{\cong} \{\Gamma_{U,df}\}$

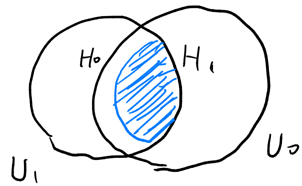
$F: (U,m) \longmapsto \Gamma_{U,df}$

F^1: Critical pt $\longmapsto$ intersection pts

$F^k = 0$, $k \geq 2$

Def. Open(X). (dg-category)

$$\begin{cases} \text{Ob}: (U, m) \\ \hom((U_0,m_0),(U_1,m_1)) = \Omega^*(\overset{/\!/\,W}{\overline{U_0}\cap U_1}, \overset{/\!/\,H_0}{\partial U_0 \cap U_1}) \\ \mu^1 = d \\ \mu^2 = \wedge, \quad \mu^k = 0,\ k \geq 3 \end{cases}$$



Now WTS $\text{Open}(X) \xrightarrow{\cong} \text{Morse}(X)$

On Object level, clearly $(U, m) \longmapsto (U, m)$

On hom level, want $\Omega^*(W, H_0) \longrightarrow \mathbb{K}\cdot \text{Cr}(W, f)$.

The idea is to use gradient flow to "localize" forms on critical pts, then show the quasi-equivalence by "Homological Perturbation theory".

Consider the homotopy of identity:

$$\begin{tikzcd} \Omega^*(W, H_0) \arrow[r, bend left, "id"] \arrow[r, bend right, "p"'] \arrow[r, phantom, "\Downarrow H"] & \Omega^*(W, H_0) \end{tikzcd}$$

given by gradient flow ψ^t of (f, g)

i.e.
$$id \cdot \omega = \omega$$
$$H_t \cdot \omega = \psi^t_* \omega$$
$$p \cdot \omega = \psi^\infty_* \omega$$

$\psi^\infty_* \omega$ may not stay in $\Omega^*(W, H_0)$. however,

$\tilde{\Psi}_\alpha \omega \in D'(W, H_0)$, the cont. linear functionals on $\Omega^*(W, H_1)$, by (Harvey–Lawson)

So we enlarge Ω^* to D':

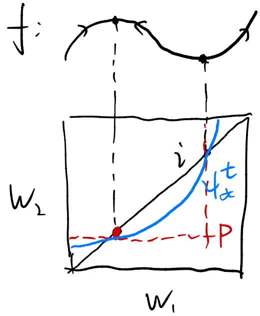
$$\begin{tikzcd} \Omega^*(W, H_0) \arrow[r, bend left, "i"] \arrow[r, bend right, "P"'] \arrow[r, phantom, "\int H"] & D'(W, H_0) \end{tikzcd}$$

s.t. $\quad i : \omega \longmapsto \left(\alpha \longmapsto \int_W \omega \wedge \alpha = \int_D p_1^*\omega \wedge p_2^*\alpha\right)$

i.e. i = Kernel diagonal $D \subset W \times W$

$\Psi_\alpha^t : \omega \longmapsto \left(\alpha \longmapsto \int_W \Psi_\alpha^t \omega \wedge \alpha = \int_{T_{\Psi^t}} p_1^*\omega \wedge p_2^*\alpha\right)$

For example, f:



(H–L): $\quad P : \omega \longmapsto \sum_{x \in Cr(f)} \omega([S_x]) \cdot [U_x]$

i.e. P = Kernel $\sum_{x \in Cr(f)} [S_x] \times [U_x]$

H = Kernel $\bigcup_{0 \le t < t_\infty} [T_{\Psi^t}]$

Also, $\quad P - i = dH + Hd \quad , \quad Pd = dP$

Still need to go from D' to Ω^*, which is just "smoothing the distribution".

Def. $R_\delta : D'(W, H_0) \longrightarrow \Omega^*(W, H_0)$

$$R_\delta(A)\cdot\beta := \int_{A\times W} \rho_\delta \wedge p_2^*\beta$$

where ρ_δ is a Thom form of D. so $R_\delta(A) \approx$ "$\rho_\delta * A$" $\in \Omega^*$

It can be shown that composed w/ R_δ makes no problems, so just denote $P_\delta = R_\delta \circ P$.

Now we have

$$\underset{(\mathrm{Open}(X))}{\Omega^*(W, H_0)} \xrightarrow{P_\delta} \underset{(\mathcal{B})}{P_\delta\Omega^*(W, H_0)} \xrightarrow{g'} \underset{(\mathrm{Morse}(X))}{\mathbb{K}\cdot cr(f)}$$

Steps 1: construct A_∞-category $\mathcal{B}$

s.t. $hom_{\mathcal{B}} = \{P_\delta\Omega^*(W, H_0)\}$, $Ob_{\mathcal{B}} = Ob_{open(X)}$

that is to define composition maps $m_k^{\mathcal{B}}$ & show A_∞-relation.

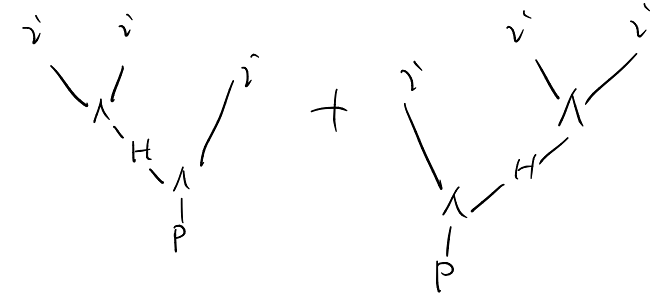
Def. $m_1^B := p\circ d\circ i$

$m_2^B := p\circ\wedge\circ(i\otimes i)$

$m_n^B := \sum_T \pm m_{n,T}$, $n\geq 3$.

where $m_{n,T}$ is composition given by tree T.

For example, m_3^B:



Where we always input through i & output through p.

for each inner edge, insert a H.

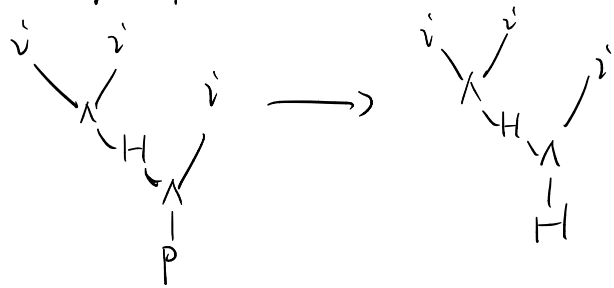
It's easy to check $\mathcal{B}$ is A_∞-category (Kontsevich & Soibelman)

Step 2: construct A_∞-functor $h : \mathcal{B} \longrightarrow \mathrm{Open}(X)$.

Def. $h^1 := i$

$$h^n := \sum_T h^n_T$$

where h^n_T is similar as m^n_T, but replace the output p by H:



Step3: construct A_∞-functor $g : \mathcal{B} \longrightarrow \mathrm{Morse}(X)$.

This is trivial, just let

$$g^1 : R_\delta\{U_x\} \longmapsto \langle x \rangle.$$

$$g^k = 0, \quad k \geq 2.$$

Also, h is quasi-equivalence since $id - i \circ p = dH - Hd$.

g is clearly quasi-equivalence.

∎