Notes on Fukaya category

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This note is a beginner version of the beginner's introduction to Fukaya category [1]. Some parts are also based on [2–4].

1 Lagrangian Floer Cohomology

In this note we assume that the background symplectic manifold (M, ω) is exact, i.e. $\omega = d\theta$. Roughly speaking, we want to study Lagrangian submanifolds in M and their intersection theory. Here we just consider exact Lagrangians L, i.e. $\theta|_L = df$ for some function $f \in C^\infty(L, \mathbb{R})$.

Example For a compact real analytic smooth manifold X, consider the cotangent bundle $M = T^*X$, then there is a canonical exact symplectic form $\omega = d\theta$, where θ is the canonical Liouville 1-form locally given by $\theta = \sum_i p_i dq_i$. Now for a submanifold $Y \subset X$, assign a defining function $m : X \to \mathbb{R}_{\geq 0}$ for $\partial Y \subset X$, then we have the standard Lagrangian:

$$L_{Y,f} = T^*_Y X + \Gamma_{df} \subset T^*X$$

where $f = \log m$ and Γ_{df} is the graph of df. Clearly $L_{Y,f}$ depends only on $f|_Y$. Since locally $\theta = \sum_i p_i dq_i$ and $p_i = \frac{\partial f}{\partial q_i}$, $\theta = dF$, where F is the pullback of f from Y to $L_{Y,f}$. Therefore $L_{Y,f}$ is exact.

1.1 Floer complex

Now we introduce the Floer complex. Roughly speaking, for each pair of Lagrangians (L_0, L_1), we define a cochain complex $CF(L_0, L_1)$, which is freely generated by intersection points of L_0 and L_1 over some field $\mathbb{K}$, together with a differential $d : CF(L_0, L_1) \to CF(L_0, L_1)[1]$.

Naturally there arise some issues: We want a well-defined Floer cohomology $HF(L_0, L_1) = \operatorname{Ker} d / \operatorname{Im} d$, so we need $d^2 = 0$; If L_0 and L_1 are not transversal, we have to perturb them by Hamiltonian isotopy so that the Floer complex is well defined; Also, we want to give $CF(L_0, L_1)$ a grading.

For now simply assume L_0, L_1 are compact and transversal, then define the Floer differential by counting J-holomorphic strips in M with boundary in L_0 and L_1 connecting p and q. Then the coefficient of q in dp is given by that number:

Definition 1 *(Floer differential)*

$$d(p) = \sum_{q \in \chi(L_0, L_1), ind([u])=1} (\#\mathcal{M}(p, q; [u], J))q, \tag{1}$$

More precisely, we are counting the space of maps $u : \mathbb{R} \times [0, 1] \to M$ which solves the Cauchy-Riemann equation $\bar{\partial}_J u = 0$:

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = 0, \tag{2}$$

$$\begin{cases} u(s, 0) \in L_0 \text{ and } u(s, 1) \in L_1, \\ \lim_{s \to +\infty} u(s, t) = p, \lim_{s \to -\infty} u(s, t) = q, \end{cases} \tag{3}$$

where J is any almost complex structure compatible with ω. Fix a homotopy class $[u] \in \pi_2(M, L_0 \cup L_1)$, then let $\#\hat{\mathcal{M}}(p, q; [u], J)$ be the moduli space of all solutions in $[u]$. Let $\#\mathcal{M}(p, q; [u], J)$ be $\#\hat{\mathcal{M}}(p, q; [u], J)$ quotiented by reparametrization of the domain, which in this case is the 1-dimensional translation of s.

The above boundary Cauchy-Riemann problem is linearized to be Fredholm. Denote the linearization of $\bar{\partial}_J$ at u by $D_{\bar{\partial}_J u}$, then the Fredholm index is defined by $ind([u]) = \dim \operatorname{Ker} D_{\bar{\partial}_J u} / \operatorname{Coker} D_{\bar{\partial}_J u}$. The dimension of moduli space $\#\hat{\mathcal{M}}(p, q; [u], J)$ is $ind([u])$ if $D_{\bar{\partial}_J u}$ is surjective everywhere on $\#\hat{\mathcal{M}}(p, q; [u], J)$. For simplicity we have to believe that for generic J this is true.

To count signed moduli spaces, orientation is necessary. Here either assume $char(\mathbb{K}) = 2$ or that L_0, L_1 are oriented and spin, then orientation is guaranteed.

1.2 Grading and Maslov index

Now it is natural to give $CF(L_0, L_1)$ gradings. To do this, try to assign to each Lagrangian L a phase function $\phi_L : L \to S^1$. If $2c_1(TM) = 0$, the bicanonical bundle $\Lambda_{\mathbb{C}}^{top} T^*M \otimes \Lambda_{\mathbb{C}}^{top} T^*M$ is trivial, then choose a nonzero section μ, define $\phi_L(p) = \frac{\mu(v_1 \wedge ... \wedge v_n \otimes v_1 \wedge ... \wedge v_n)}{||\mu(v_1 \wedge ... \wedge v_n \otimes v_1 \wedge ... \wedge v_n)||}$. It is easy to check the independence on basis of T_pL.

We still need to lift the phase function $\phi_L : L \to S^1$ to $\widetilde{\phi}_L : L \to \mathbb{R}$, so require that the Maslov class $[\phi_L] \in [L, S^1]$ vanishes.

Assume $2c_1(TM) = 0$ and all Maslov classes vanish, then we can define:

Definition 2

$$deg(p) = \widetilde{\phi}_{L_0}(p) - \widetilde{\phi}_{L_1}(p) + \lambda(p, L_0, L_1), \tag{4}$$

where $\lambda(p, L_0, L_1)$ accounts for the difference between T_pL_0 and T_pL_1, lifted to $\mathbb{R}$ by a short path in $LGr(T_pM)$. Then the Maslov index is defines as:

$$ind(u) = deg(q) - deg(p), \tag{5}$$

where u is any J-holomorphic strip connecting p to q.

For standard Lagrangians $L_{Y,f} = T_Y^*X + \Gamma_{df} \subset T^*X$ in T^*X, grading is canonically well-defined:

Proposition 1 *The bicanonical bundle $\Lambda_{\mathbb{C}}^{top} T^*(T^*X) \otimes \Lambda_{\mathbb{C}}^{top} T^*(T^*X)$ is canonically trivial; The Maslov class $[\phi_{L_{Y,f}}] = 0$. Therefore, there is a canonical grading of $L_{Y,f}$.*

It turns out that Maslov index is the same as Fredholm index:

Proposition 2 *(A relative version of Riemann-Roch)*

$$ind(u) = ind_{Fredholm}(D_u), \tag{6}$$

1.3 Compactness

Compactness of $\mathcal{M}(p, q; [u], J)$ is given by Gromov compactness theorem. First assume a uniform energy bound. There are three possible limit behavior: strip breaking, disc bubbling and sphere bubbling. By assumption that M and L are both exact, the latter two possibilities are eliminated.

Now by some gluing statement, we can say broken strips correspond to the boundary of moduli space of strips with $ind([u]) = 2$:

$$\partial\overline{\mathcal{M}}(p, q; [u], J) = \coprod_{[u']+[u'']=[u], r\in\chi(L_0,L_1)} (\mathcal{M}(p, r; [u], J) \times \mathcal{M}(r, q; [u], J)), \quad (7)$$

from which it is easy to see $d^2 = 0$. Moreover, we have

Theorem 1 *(Floer) The Floer differnetial is well-defined: $d^2 = 0$; The Floer cohomology $HF(L_0, L_1)$ is independent of J and Hamiltonian isotopies of L_0 and L_1.*

1.4 Transversality

In case L_0, L_1 do not intersect transversally (for example $L_0 = L_1$), try to perturb one of them by (time-dependent) Hamiltonian isotopy, i.e. choose a generic Hamiltonian $H \in C^\infty([0, 1] \times M, \mathbb{R})$ and let $CF(L_0, L_1)$ be generated by points in $L_0 \cap (\phi_H^1)^{-1}L_1$.

Equivalently, we may perturb the Cauchy-Riemann equation instead of Lagrangians:

$$\frac{\partial u}{\partial s} + J(t, u)\left(\frac{\partial u}{\partial t} - X_H(t, u)\right) = 0, \quad (8)$$

$$\begin{cases} u(s, 0) \in L_0 \text{ and } u(s, 1) \in L_1, \\ \lim_{s\to+\infty} u(s, t) \text{ and } \lim_{s\to-\infty} u(s, t) \text{ are flow of } X_H \text{ from } L_0 \text{ to } L_1, \end{cases} \quad (9)$$

2 Fukaya category

2.1 Composition map

We start the discussion of Fukaya category by introducing higher composition maps:

Definition 3 *(Higher operations)*

$$\begin{aligned} &\mu^k : CF(L_{k-1}, L_k) \otimes ... \otimes CF(L_1, L_2) \otimes CF(L_0, L_1) \to CF(L_0, L_k)[2 - k], \\ &\mu^k(p_k, ..., p_1) = \sum_{ind([u])=2-k, q\in\chi(L_0,L_k)} (\#\mathcal{M}(p_1, ..., p_k, q; [u], J))q, \end{aligned} \quad (10)$$

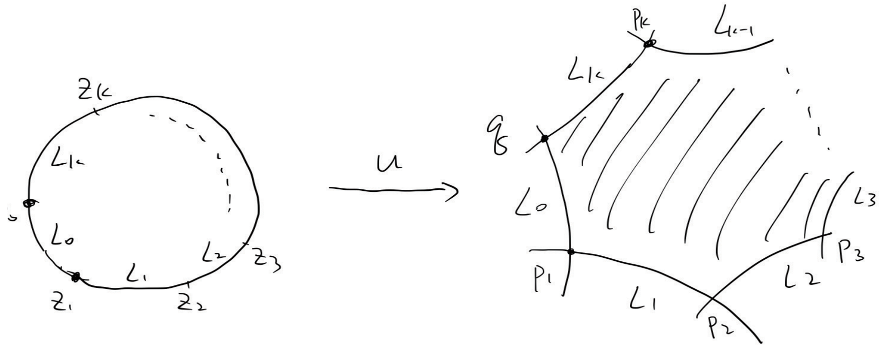

Figure 1: $\mathcal{M}(p_1, ..., p_k, q; [u], J)$

where $\mathcal{M}(p_1, ..., p_k, q; [u], J)$ *is the moduli space of k-pointed J-holomorphic disk with boundary in* $L_0, ..., L_k$ *as shown in Figure 1, quotiented by reparametrization. The Maslov index in the sum comes from*

$$\dim \mathcal{M}(p_1, ..., p_k, q; [u], J) = k - 2 + ind([u]) = k - 2 + deg(q) - \sum_{i=1}^{k} deg(p_i), \tag{11}$$

Note that $\mu^1 = d$ and $(\mu^1)^2 = d^2 = 0$. This can be generalized to $k \geq 1$, which is called A_∞-relation. Still we consider the limit curve (boundary) of every 1-dim moduli space $\mathcal{M}(p_1, ..., p_k, q; [u], J)$. Similar as the case of $k = 1$, the limit curve cannot be disc bubbling or sphere bubbling, so the only possible behavior is nodal discs spliting the k-pointed disc. The signed sum of all these limit curves is zero, which implies

Proposition 3 *(A_∞-relations)*

$$\sum_{l=1}^{k} \sum_{j=0}^{k-l} (-1)^* \mu^{k+1-l}(p_k, ..., p_{j+l+1}, \mu^l(p_{j+l}, ...p_{j+1}), p_j, ..., p_1) = 0, \tag{12}$$

where $* = j + deg(p_1) + ... + deg(p_j)$.

2.2 Definition of $Fuk(M,\omega)$

Definition 4 *Let (M,ω) be exact with $2c_1(TM)=0$.*

$$Ob(Fuk(M,\omega)) = \{\textit{exact Langrangian } L \textit{ with spin structure and graded lift}\}, \tag{13}$$

For each pair of objects L, L' which need not be transversal, choose time-dependent perturbation $H_{L,L'}$ and $J_{L,L'}$. Then for every $L_0,...,L_k$ and $J-$holomorphic discs, choose H and L compatible with each pair's near each end. Such perturbation exists.

$$hom(L,L') = CF(L,L';H_{L,L'},J_{L,L'}). \tag{14}$$

Then $Fuk(M,\omega)$ is the A_∞-category with composition maps given by Definition 3.

Usually a triangulated category is better so that we can talk about generators and mapping cones. Therefore we embed $Fuk(M,\omega)$ into a larger one $TwFuk(M,\omega)$, which contains twisted complexes of objects in $Fuk(M,\omega)$. Precisely,

Definition 5 *$TwFuk(M,\omega)$*

$$Ob(TwFuk(M,\omega)) = (E,\delta^E), \tag{15}$$

where $E = \bigoplus_{i=1}^{N} L_i[k_i]$. $\delta^E \in End^1(E)$ is lower trangular differential s.t. $\sum_{k\geq 1}\mu^k(\delta^E,...,\delta^E)=0$.

A degree d morphism a in $hom(E,E')$ is just the direct sum of all $a_{ij} \in hom^{d+k'_j-k_i}(E_i,E'_j)$.

Composition maps are given by

$$\mu^k_{Tw}(a_k,...,a_1) = \sum_{j_0,...,j_k\geq 0}\mu^{k+j_0+...+j_k}(\delta^k,...,\delta^k,a_k,...,\delta^1,...,\delta^1,a_1,\delta^0,...,\delta^0). \tag{16}$$

Definition 6 *(mapping cones) If $f\in hom^0(E,E')$ is closed, then*

$$\begin{tikzcd}
(E,\delta) \arrow[rr, "f"] & & (E',\delta') \arrow[d] \\
& \left(E[1]\oplus E', \begin{pmatrix}\delta & 0\\ f & \delta'\end{pmatrix}\right) \arrow[ul, "(1)"] &
\end{tikzcd} \tag{17}$$

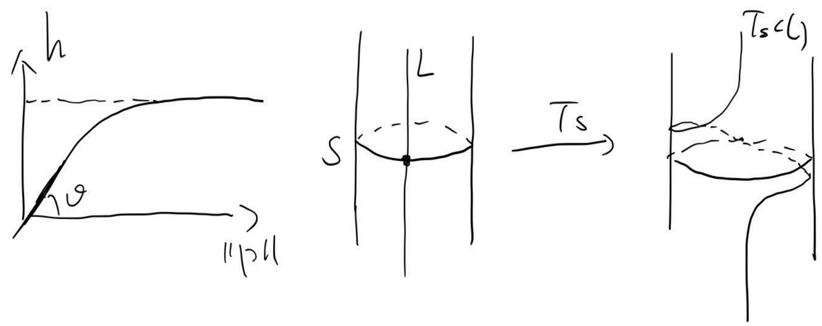

Figure 2: Dehn twist

Example (Dehn twists) Given a Lagrangian sphere S in (M, ω), by Weinstein, a neighbourhood of S in M is just like T^*S. Now perform Dehn twists by a Hamiltonian flow $H(p, q) = h(||p||)$ in the complement of zero section as Figure 2.

Then $\tau_S(L)$ is the mapping cone:

Theorem 2 *(Seidel)*

$$\begin{tikzcd} HF^*(S,L)\otimes S \arrow[r, "ev"] & L \arrow[d] \\ & \tau_S(L) \arrow[ul, "(1)"] \end{tikzcd} \tag{18}$$

2.3 Non-compact Lagrangians and perturbation

Here gives an example on dealing with non-compact Lagrangians by [3]. Let $U \subset X$ be open, fix a defining function m for $X \setminus U$, let $f = \log m$. Let L be standard Lagrangian given by the graph df. Fix stratification $\mathcal{S} = \{S_\alpha\}$ of X, and let $\Lambda_{\mathcal{S}} = \cup_\alpha T^*_{S_\alpha} X \subset T^*X$ be the corresponding conical Lagrangian.

If L and $\Lambda_{\mathcal{S}}$ intersect at infinity, then their intersections may be non-compact. To restrict all intersections inside a compact region, need to perturb L s.t. they are separated at infinity.

Proposition 4 *(Nadler-Zaslow) There exist $\eta > 0$ and $\delta > 0$ such that for all $\delta' \in (0, \delta]$, the normalized geodesic flow satisfies*

$$\gamma_{\delta'}(\bar{L}_{m \leq \eta}) \cap \bar{\Lambda}_{\mathcal{S}} = \emptyset, \tag{19}$$

where $\gamma_{\delta'}$ is the flow of Hamiltonian $H(x, \xi) = |\xi|$ on $T^*X \setminus X$.

References

[1] Denis Auroux. A beginner's introduction to fukaya categories. In *Contact and symplectic topology*, pages 85–136. Springer, 2014.

[2] Masaki Kashiwara and Pierre Schapira. *Sheaves on Manifolds: With a Short History.Les débuts de la théorie des faisceaux. By Christian Houzel*, volume 292. Springer Science & Business Media, 2013.

[3] David Nadler and Eric Zaslow. Constructible sheaves and the fukaya category. *Journal of the American Mathematical Society*, 22(1):233–286, 2009.

[4] Paul Seidel. *Fukaya categories and Picard-Lefschetz theory*, volume 10. European Mathematical Society, 2008.