Review of A_∞ Categories, Twisted Complexes, and Triangles

Eilon Reisin-Tzur

May 01, 2019

Definition: An $\boldsymbol{A_\infty}$ **category** $\mathcal{A}$ consists of:

1. A set of objects $\mathrm{Ob}\,\mathcal{A}$
2. A graded vector space $\hom_{\mathcal{A}}(X_0, X_1)$ for any $X_0, X_1 \in \mathrm{Ob}\,\mathcal{A}$
3. Composition maps of every order $d \geq 1$

$$\mu_A^d : \hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0, X_1) \to \hom_{\mathcal{A}}(X_0, X_d)\,[2-d]$$

satisfying the following A_∞ associativity equations:

$$\sum_{m,n} (-1)^{\heartsuit_n} \mu_A^{d-m+1}\left(a_d, \ldots, a_{n+m+1}, \mu_{\mathcal{A}}^m\left(a_{n+m}, \ldots, a_{n+1}\right), a_n, \ldots, a_1\right) = 0,$$

where $\heartsuit_n = |a_1| + \ldots + |a_n| - n$ and the sum is over all possible terms $(1 \leq m \leq d, 0 \leq n \leq d - m)$.

(Note: [k] corresponds to shifting the grading down by k.)

Let's look at these composition maps (up to signs):
If $d = 1$, then we have that $\mu^1(\mu^1(a_1)) = 0$. In fact, we will form the cohomological category of $\mathcal{A}$, where μ_1 will be the differential.
If $d = 2$, we get that $\mu^2(\mu^1(a_2), a_1) + \mu^2(a_2, \mu^1(a_1)) - \mu^1(\mu^2(a_2, a_1)) = 0$. The choices of signs here to suggest more directly that μ^2 is a chain map.
For $d = 3$ and generalizations to higher composition maps, we view the associativity equation as all the way to break up our input into two steps as shown in Figure 1.

Given an A_∞ category $\mathcal{A}$, we can define the cohomological category $\mathrm{H}(\mathcal{A})$, where $\mathrm{Ob}\,\mathrm{H}(\mathcal{A}) = \mathrm{Ob}\,\mathcal{A}$ and the morphisms are the cohomology groups $\mathrm{H}(\hom_{\mathcal{A}}(X_0, X_1), \mu_{\mathcal{A}}^1)$. Composition is given by $[a_2]\dot{[}a_1] = (-1)^{|a_1|}[\mu_{\mathcal{A}}^2(a_2, a_1)]$.

Examples:

- If $\mathcal{A}$ has only one object, then $\mathcal{A}$ is an A_∞ algebra
- If $\mu_{\mathcal{A}}^d = 0$ for all $d > 2$, then $\mathcal{A}$ is a dg-category.

$\boldsymbol{A_\infty}$ **Functors:**
An A_∞ functor between A_∞ categories $\mathcal{A}$ and $\mathcal{B}$ consists of a map $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ with maps of every order $d \geq 1$

$$\mathcal{F}^d : \hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0, X_1) \longrightarrow \hom_{\mathcal{B}}(\mathcal{F}X_0, \mathcal{F}X_d)\,[1-d]$$

satisfying

$$\begin{aligned} &\textstyle\sum_r \sum_{s_1,\ldots,s_r} \mu_{\mathcal{B}}^r\left(\mathcal{F}^{s_r}\left(a_d, \ldots, a_{d-s_r+1}\right), \ldots, \mathcal{F}^{s_1}\left(a_{s_1}, \ldots, a_1\right)\right) \\ &\quad = \textstyle\sum_{m,n} (-1)^{\heartsuit_n} \mathcal{F}^{d-m+1}\left(a_d, \ldots, a_{n+m+1}, \mu_{\mathcal{A}}^m\left(a_{n+m}, \ldots, a_{n+1}\right), a_n, \ldots, a_1\right) \end{aligned}$$

where the left-hand sum is over all $r \geq 1$ and partitions $s_1 + \ldots + s_r = d$.

Note that on both sides of the equation, we are only applying a composition map once in each term.

Let's have a look at what these relations actually mean (up to sign) when d is small:

If $d=1$, then $\mu^1_{\mathcal{B}}(\mathcal{F}^1(a_1)) = \mathcal{F}^1(\mu^1_{\mathcal{A}}(a_1))$, meaning that $\mathcal{F}^1$ commutes with the differentials up to sign.

If $d=2$, then after some rearrangement and suggestive sign choices we have

$$\mu^2(\mathcal{F}^1(a_2), \mathcal{F}^1(a_1)) - \mathcal{F}^1(\mu^2(a_2,a_1)) = \mu^1(\mathcal{F}^2(a_2,a_1)) - \mathcal{F}^2(\mu^1(a_2),a_1) - \mathcal{F}^2(a_2,\mu^1(a_1))$$

. The subscripts on the μ^d have been dropped since they can be inferred by the arguments of the map. The left side can be interpreted as a measure of how $\mathcal{F}^1$ and μ^2 fail to commute while the right side is the failure of μ^1 to be a derivation. So in some sense, this relation is saying that the failure to commute is equal to the failure to be a derivation (after ignoring information about the signs).

For the case where $d=3$, take a look at Figure 2, where shading corresponds to the $\mathcal{F}^d$ and the μ^d are represented as in Figure 1.

An A_∞ functor $\mathcal{F}$ induces a functor $\mathrm{H}(\mathcal{F}) : \mathrm{H}\,\mathcal{A} \to \mathrm{H}\,\mathcal{B}$, where $[a] \mapsto [\mathcal{F}^1(a)]$.

Lastly, composition of functors is strictly associate, given by

$$(\mathcal{G} \circ \mathcal{F})^d (a_d, \ldots, a_1) = \sum_r \sum_{s_1,\ldots,s_r} \mathcal{G}^r \left(\mathcal{F}^{s_r}\left(a_d, \ldots, a_{d-s_r+1}\right), \ldots, \mathcal{F}^{s_1}\left(a_{s_1}, \ldots, a_1\right)\right).$$

Twisted Complexes:

Definition: A **twisted complex** (E, δ^E) consists of:

- a formal direct sum $E = \bigoplus_{i=1}^{N} E_i[k_i]$ of shifted objects of $\mathcal{A}$
- a strictly lower triangular differential $\delta^E \in End(E)$ satisfying $\sum_{k \geq 1} \mu^k(\delta^E, ..., \delta^E) = 0$, where δ^E is a collection of maps $\delta^E_{ij} \in \hom^{k_j - k_i + 1}(E_i, E_j)$, $i < j$

A degree d morphism of two twisted complexes (E, δ^E), $(E' \delta^{E'})$ is a degree d map between the direct sums. That is, $a \in \hom^d(E, E')$ is a collection of maps $a_{ij} \in \hom^{d + k'_j - k_i}(E_i, E'_j)$.

Now our goal is to make an A_∞ category where the objects are the twisted complexes.

Given twisted complexes $(E_0, \delta^0), ..., (E_k, \delta^k)$ and morphisms $a_i \in \hom(E_{i-1}, E_i)$, we set

$$\mu^k_{\mathrm{Tw}}\left(a_k, \ldots, a_1\right) = \sum_{j_0, \ldots, j_k \geq 0} \mu^{k+j0+\cdots+j_k} \left(\underbrace{\delta^k, \ldots, \delta^k}_{j_k}, a_k, \ldots, \underbrace{\delta^1}_{j_1}, a_1, \underbrace{\delta^0, \ldots, \delta^0}_{j_0} \right).$$

This defines the composition maps needed to make the set of all twisted complexes formed by objects of $\mathcal{A}$ into an A_∞ category denoted by $\mathrm{Tw}\,\mathcal{A}$.

Proposition: $\mathcal{A}$ embeds fully faithfully into $\mathrm{Tw}\,\mathcal{A}$ by viewing each object in $\mathcal{A}$ as its own twisted complex in grading 0.

Definition: Given twisted complexes (E, δ^E), $(E' \delta^{E'}) \in \mathrm{Tw}\,\mathcal{A}$, and a closed morphism $f \in hom^0(E, E')$ (meaning $\mu^1_{\mathrm{Tw}}(f) = 0$), the **abstract mapping cone of f** is the twisted complex

$$\mathrm{Cone}(f) = \left(E[1] \oplus E', \begin{pmatrix} \delta & 0 \\ f & \delta' \end{pmatrix} \right).$$

Note: This mimics mapping cones in the category of chain complexes.

This gives an exact triangle in the usual sense,

$$\begin{tikzcd} A \arrow[rr, "f"] & & B \arrow[dl, "i"] \\ & Cone(f) \arrow[ul, "p", "{[1]}"'] & \end{tikzcd}$$

where i is the inclusion of $B \hookrightarrow A[1] \oplus B$ and p is the projection $A[1] \oplus B \to A[1]$.

Definition: An **exact triangle** in an A_∞ category $\mathcal{A}$ consists of $A, B, C \in \operatorname{Ob}\mathcal{A}$, closed morphisms $f \in \hom^0(A, B), g \in \hom^0(B, C), h \in \hom^1(C, A)$ such that C is quasi-isomorphic to $Cone(f) \in \operatorname{Tw}\mathcal{A}$.

C quasi-isomorphic to $Cone(f)$ means they are isomorphic in the cohomology category $\mathrm{H}(\operatorname{Tw}\mathcal{A})$ when C is viewed as an object in the twisted category.

Exactness refers to the composition of any two of the maps being exact in the usual sense.

The exact triangle above is often represented by the following diagram

$$\begin{tikzcd} A \arrow[rr, "f"] & & B \arrow[dl, "g"] \\ & C \arrow[ul, "h", "{[1]}"'] & \end{tikzcd}$$

Given the exact triangle above and any test object $T \in \mathrm{H}(\mathcal{A})$, we get a long exact sequence

$$\to \mathrm{H}^i \hom(T, A) \xrightarrow{f} \mathrm{H}^i \hom(T, B) \xrightarrow{g} \mathrm{H}^i \hom(T, C) \xrightarrow{h} \mathrm{H}^{i+1} \hom(T, A) \to$$

Definition: We say that an A_∞ category $\mathcal{A}$ is **triangulated** if every morphism $f : A \to B$ can be completed to an exact triangle. That is, mapping cones exist in $\mathcal{A}$.

Proposition: $\operatorname{Tw}\mathcal{A}$ is triangulated.
Proof: This is essentially by design and the proof is identical to that for chain complexes.

Lemma: An A_∞ category $\mathcal{A}$ is triangulated if and only if the embedding $\mathcal{A} \hookrightarrow \operatorname{Tw}\mathcal{A}$ is a quasi-equivalence.

We now give a slightly different view of exact triangles in an A_∞ category. An triangle is a diagram in $\mathrm{H}(\mathcal{A})$

$$\begin{tikzcd} Y_0 \arrow[rr, "{[c_1]}"] & & Y_1 \arrow[dl, "{[c_2]}"] \\ & Y_2 \arrow[ul, "{[c_3]}", "{[1]}"'] & \end{tikzcd}$$

We must take a little detour and define identity morphisms in an A_∞ category.

Definition: Morphisms e_{X_i} are **identity morphisms** if $\mu^1(e_{X_i}) = 0$, $\mu^2(a, e_{X_0}) = a = (-1)^{|a|}\mu^2(e_{X_1}, a)$, and $\mu^d(a_{d-1}, ..., a_{n+1}, e_{X_n}, a_n, ..., a_1) = 0$ for $d > 2$.

Proposition: Let $\mathcal{D}$ be an A_∞ category with $Z_0, Z_1, Z_2 \in \operatorname{Ob}\mathcal{D}$ such that

- $\hom(Z_k, Z_k) = \langle e_{Z_k} \rangle$
- $\hom(Z_0, Z_1) = \langle x_1 \rangle$ where $|x_1| = 0$, $\hom(Z_1, Z_2) = \langle x_2 \rangle$ where $|x_2| = 0$, and $\hom(Z_2, Z_0) = \langle x_3 \rangle$ where $|x_3| = 1$
- $\hom(Z_2, Z_1) = \hom(Z_1, Z_0) = \hom(Z_0, Z_2) = 0$
- $\mu^3(x_3, x_2, x_1) = e_{Z_0}$, $\mu^3(x_1, x_3, x_2) = e_{Z_1}$, $\mu^3(x_2, x_1, x_3) = e_{Z_2}$

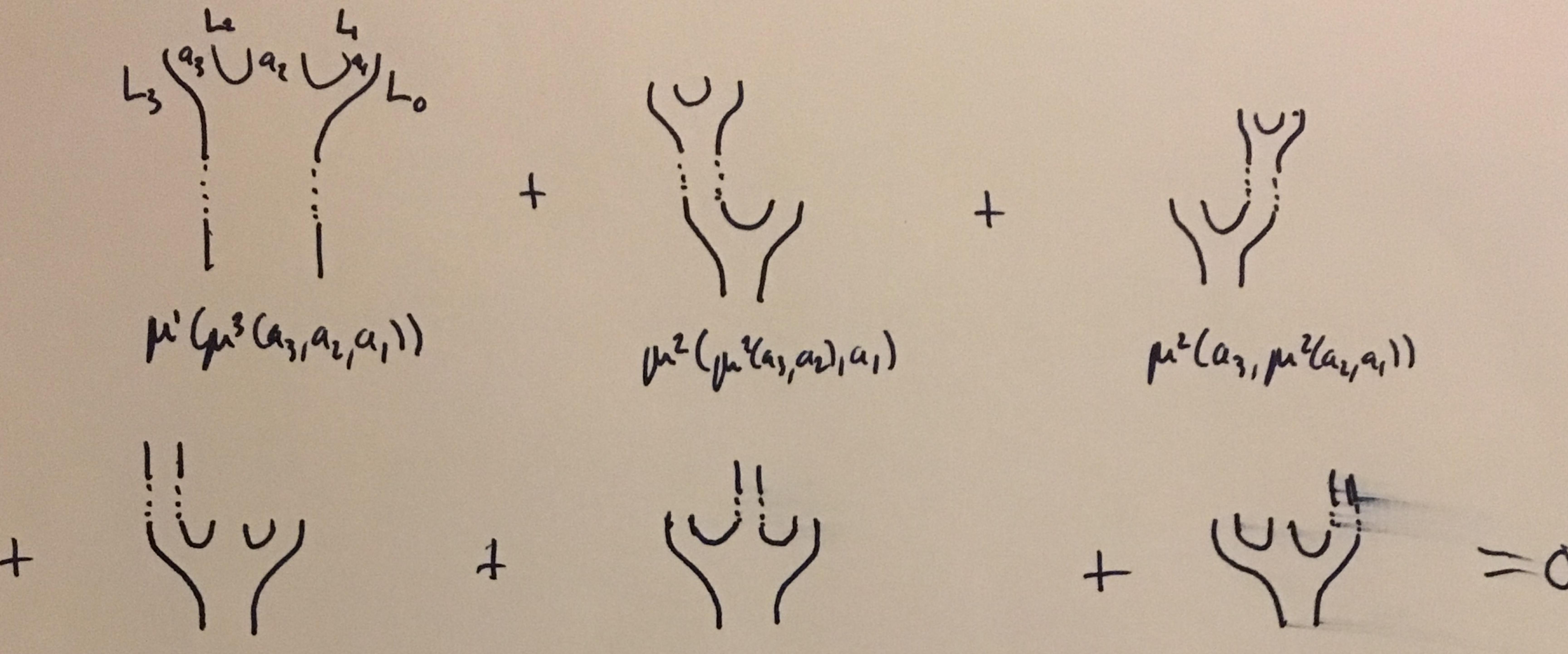
Then a triangle in $\mathrm{H}(\mathcal{A})$ is exact if and only if there exists an A_∞ functor $\mathcal{F} : \mathcal{D} \to \mathcal{A}$ such that $\mathcal{F}(Z_k) = Y_k$ and $[\mathcal{F}^1(X_k)] = [c_k]$.

References

1. Denis Auroux. A beginner's introduction to fukaya categories. In Contact and symplectic topology, pages $85-136$. Springer, 2014 .

2. Paul Seidel. Fukaya categories and Picard-Lefschetz theory, volume 10 . European Mathematical Society, 2008 .

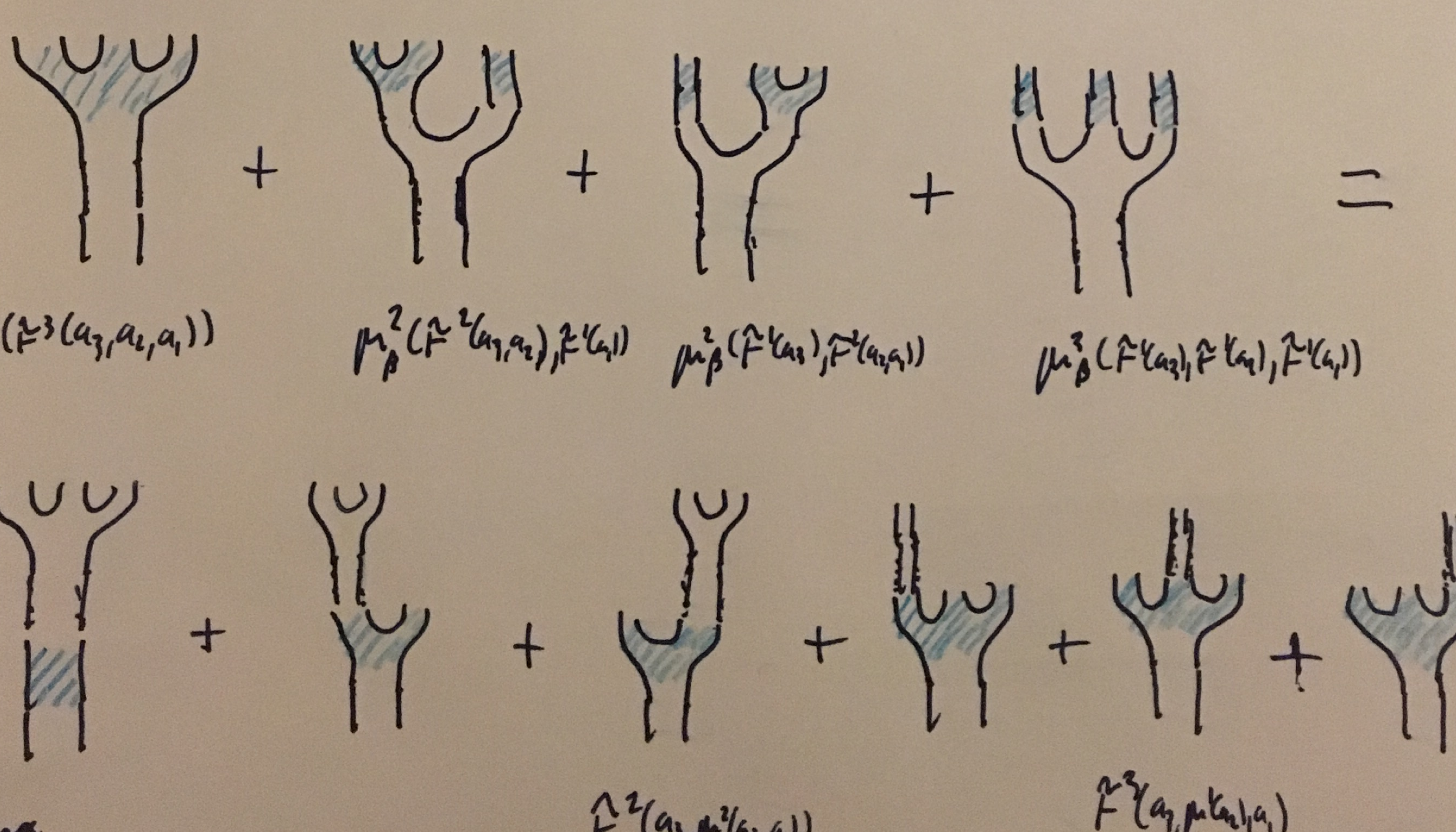
Figure 1:



$\mu^1(\mu^3(a_3,a_2,a_1))$ $+$ $\mu^2(\mu^2(a_3,a_2),a_1)$ $+$ $\mu^2(a_3,\mu^2(a_2,a_1))$

$+$ $\mu^3(\mu^1(a_3),a_2,a_1)$ $+$ $\mu^3(a_3,\mu^1(a_2),a_1)$ $+$ $\mu^3(a_3,a_2,\mu^1(a_1))$ $=0$

Figure 2:



$\mu^1_\beta(\tilde{F}^3(a_3,a_2,a_1))$ $+$ $\mu^2_\beta(\tilde{F}^2(a_3,a_2),\tilde{F}^1(a_1))$ $+$ $\mu^2_\beta(\tilde{F}^1(a_3),\tilde{F}^2(a_2,a_1))$ $+$ $\mu^3_\beta(\tilde{F}^1(a_3),\tilde{F}^1(a_2),\tilde{F}^1(a_1))$ $=$

$\tilde{F}^1(\mu^3(a_3,a_2,a_1))$ $+$ $\tilde{F}^2(\mu^2(a_3,a_2),a_1)$ $+$ $\tilde{F}^2(a_3,\mu^2(a_2,a_1))$ $+$ $\tilde{F}^3(\mu^1(a_3),a_2,a_1)$ $+$ $\tilde{F}^3(a_3,\mu^1(a_2),a_1)$ $+$ $\tilde{F}^3(a_3,a_2,\mu^1(a$