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Micro-support - Seminar

References: 'an intro to symplectic topology through sheaf theory' - Viterbo, KS

I. Definition of microsupport
  A. X a manifold, $D^b(X)$ bounded derived category of sheaves (of A-modules) over X.
  B. Let $j : U \to X$ be an open embedding, $i : Z \to X$ the closed embedding of $Z = X \backslash U$. Let $\mathcal{F} \in D^b(X)$.
    i. want to make object equal to $\mathcal{F}$ on either U or Z, 0 elsewhere. so, pullback and apply extension by zero functor.
    ii. let $\mathcal{F}_Z = Ri_* i^{-1} A_X$ denote the sheaf that agrees with $\mathcal{F}$ on Z and is zero elsewhere. (for a generally locally closed embedding, we have $\mathcal{F}_Z = Ri_! i^{-1}\mathcal{F}$, but when i is closed $i_* = i_!$. also, $f_!$ is extension by zero functor for a locally closed embedding)
    iii. let $\mathcal{F}_U = Rj_! j^* \mathcal{F}$ be the sheaf that agrees with $\mathcal{F}$ on U and is zero elsewhere. have to use $j_!$, consider $U = D^2 - 0$. since j is an open embedding $j^! = j^*$
    iv. There are triangles: $i_! i^! \to \mathrm{id} \to j_* j^*$ and $j_! j^! \to \mathrm{id} \to i_* i^*$, yielding LES in relative cohomology. specifically, the second one gives $\mathcal{F}_U \to \mathcal{F} \to \mathcal{F}_Z$
  C. $Z \subset X$ closed, let Γ_Z denote the functor 'sections with support in Z', so that we have an exact sequence $\Gamma_Z(\mathcal{F}) \to \mathcal{F} \to \mathcal{F}_U = j_! j^* \mathcal{F}$ of sheaves on X. Let $R\Gamma_Z$ be the right derived functor.
    i. on sections, $\Gamma_Z \mathcal{F}(V) = \{s \in \mathcal{F}(V) | \operatorname{supp}(s) \subset Z\} = \ker(\mathcal{F}(V) \to \mathcal{F}(V \cap U))$
    ii. Γ_Z is left exact because restriction to U is, the derived functor $R\Gamma_Z$ is relative cohomology, cohomology with supports in Z, local cohomology.
    iii. $i^! = i^* R\Gamma_Z$
    iv. $H^*(X, i_! i^! \mathcal{F}) = H^*(Z, i^! \mathcal{F}) = H^*(X, U; \mathcal{F})$
    v. $\Gamma_Z(\mathcal{F}) = \mathrm{Hom}(A_Z, \mathcal{F})$
  D. For $\mathcal{F}^\bullet \in D^b(X)$, $SS(\mathcal{F}) = \overline{\{(x,p) \in T^*X | (x,p) \text{ satisfies } (*)\}}$
    i. (*) there exists a function $\varphi : X \to \mathbb{R}$ such that $\varphi(x) = 0$, $d\varphi(x) = p$, and $R\Gamma_{\{x | \varphi(x) \geq 0\}}(\mathcal{F})_x \neq 0$.
    ii. vanishing of $R\Gamma_Z$ is equivalent to restriction $H^j(U, \mathcal{F}) \to H^j(U \backslash Z, \mathcal{F})$ being an isomorphism, at least for sheaves

II. properties
  A. $SS(\mathcal{F})$ is a conic subset of T^*X (closed under multiplication by $\mathbb{R}_{\geq 0}$)
  B. $SS(\mathcal{F}) \cap 0_X = \operatorname{supp}(\mathcal{F})$ (taking $\varphi = 0$)
  C. $(x,p) \in SS(\mathcal{F})$ depends only on $\mathcal{F}$ near x
  D. assume $\mathcal{F}$ is a single sheaf in degree 0, then $R\Gamma_{\{x|\varphi(x)\geq 0\}}(\mathcal{F})_x = 0$ is equivalent to the restriction map $\lim_{x \in U} H^j(U; \mathcal{F}) \to \lim_{x \in U} H^j(U \cap \{\varphi < 0\}; \mathcal{F})$ is an isomorphism for all j (from the LES associated to the defining SES)
    i. $R\Gamma^j_{\{x|\varphi(x)\geq 0\}}(\mathcal{F}) = \ker(H^j(U; \mathcal{F}) \to H^j(U \cap \{\varphi < 0\}; \mathcal{F}))$
  E. If $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to +1$ is a distinguished triangle then $SS(\mathcal{F}_i) \subset SS(\mathcal{F}_j) \cup SS(\mathcal{F}_k)$ and $SS(\mathcal{F}_j) \Delta SS(\mathcal{F}_i) \subset SS(\mathcal{F}_k)$ for any $i, j, k = 1, 2, 3$

i. apply $R\Gamma_Z$ to the triangle, $Z = \{y|\psi(y) \geq 0\}$. if two of the terms in the LES vanish, so does the third. If one vanishes, the other two are isomorphic, so one vanishes iff the other does.

F. $SS(\mathcal{F}) \subset \cup_j SS(\mathcal{H}^j(\mathcal{F}))$ (reverse inclusion false in general)

i. apply the spectral sequence from $R^p\Gamma_Z(H^q(\mathcal{F})) \to R^{p+q}\Gamma_Z(\mathcal{F})$. if first thing vanishes then so does output

III. examples

A. $SS(\mathcal{F}) = \emptyset$ iff $\mathcal{F}$ is equivalent to the 0 sheaf (has exact stalks)

B. $SS(A_X) = 0_X$, $= 0_X$ for any locally constant sheaf. all of the restriction maps are isos (except to $\emptyset$)

C. warm-up calc: Let $Z = [0, \infty)$ and $U = (-\infty, 0)$. What is $H^*(\mathbb{R}; A_Z)$? $H^*(\mathbb{R}; A_U)$?

i. A: $H^0(\mathbb{R}; A_Z) = A$, just pushfwd, so get cohomology of Z.

ii. $H^1(\mathbb{R}; A_U) = A$, A_U uses $j_!$. In fact, for any locally closed embedding j and i the inclusion of the complement, $H^*(X, U; \mathcal{F}) = H^*(X; j_!j^!\mathcal{F})$. So, we get relative cohomology $\mathbb{R}$ rel U

D. What is $SS(A_Z) \cap T_0^*\mathbb{R}$?

i. For a small nbhd V of 0, consider the restriction map $H^j(V; A_Z) \to H^j(V \cap \{x < 0\}; A_Z)$. it's $A \to 0$. has kernel, so $(0, dx) \in SS(A_Z)$.

ii. replace x by $-x$, then the restriction map is identity $A \to A$, no kernel, so $(0, -dx) \notin SS(A_Z)$.

E. Similarly for U

i. $H^j(V; A_U) \to H^j(V \cap \{x < 0\}; A_U)$ is A to A, $(x, dx) \notin SS(A_U)$.

ii. $A_U(V) \to A_U(V \cap \{x > 0\})$

F. Consider $\mathbb{R}$ stratified with points at -1, 1

G. smooth region: $U \subset X$ open, smooth region with smooth boundary ∂U. Let ν denote the exterior normal vector. Let A_U be constant sheaf on U. Then $SS(A_U) = \{(x, p)|x \in U, p = 0, \text{ or } x \in \partial U, p = \lambda\nu(v), \lambda > 0\}$

i.

H. Let Z be locally closed, A_Z the constant sheaf on Z. E a f.d. vector space, γ closed convex cone with vertex at 0, let $\gamma^\circ = \{\theta \in E^*|\theta(v) \geq 0, v \in \gamma\}$.

I. $SS(A_\gamma) \cap \pi^{-1}(0) = \gamma^\circ$

i. uses microlocal cut-off lemma, γ-topology

J. X a manifold, M closed submanifold, then $SS(A_M) = T_M^*X$

i. everything is local, reduce to X a vector space, M a subspace. then follows from previous

K. Assume φ has $d\varphi \neq 0$ on $\{\varphi(x) = 0\}$. Then $SS(A_{\{\varphi(x) \geq 0\}}) = \{(x, \lambda d\varphi)|\lambda\varphi(x) = 0, \lambda \geq 0, \varphi(x) \geq 0\}$ and $SS(A_{\{\varphi(x) > 0\}}) = \{(x, \lambda d\varphi)|\lambda\varphi(x) = 0, \lambda \leq 0, \varphi(x) \geq 0\}$

i. choose coordinates such that $\{\varphi(x) \geq 0\}$ is a closed half-space, then follows from earlier result.

ii. the open case follow from exact sequence $A_{\{\varphi(x) > 0\}} \to A_X \to A_{\{\varphi(x) \leq 0\}}$

IV. operations (skip)

A. Lagrangian correspondences: Lagrangians $\Lambda \subset T^*X \times T^*Y$, induces correspondence from T^*X to T^*Y by:

i. let $K = \Delta_{T^*X} \times T^*Y$ be a coisotropic and take $C \subset T^*X$. then $\Lambda \circ C =$ symp reduction of $C \times \Lambda \cap K$.

B. let $\pi_Y : T^*X \times T^*Y \to T^*Y$ be projection. Let $\Lambda_f = \{(x, \theta, y, \eta)|y = f(x), \theta = \eta \circ df\}$ be the Lagrangian relation associated to f.

C. Take $f : X \to Y$ a proper map on $\mathrm{supp}(\mathcal{F})$. Then $SS(Rf_*(\mathcal{F})) \subset \pi_Y(df)^{-1}(SS(\mathcal{F})) = \Lambda_f \circ SS(\mathcal{F})$, equality if f a closed embedding. similarly for $Rf_!$. For f a submersion, $SS(f^{-1}\mathcal{G}) = df(\pi_Y^{-1}(SS(\mathcal{G}))) = \Lambda_f^{-1} \circ SS(\mathcal{G})$.

   i. Take $\psi : Y \to \mathbb{R}$ with $\psi(f(x)) = 0$ and $p = d\psi(f(x))df(x)$. if $(x, p) \notin SS(\mathcal{F})$ for all $x \in f^{-1}(y)$, then $R\Gamma_{\{\psi \circ f \geq 0\}}(\mathcal{F})|_{f^{-1}(y)} = 0$.

   ii. Note that $\Gamma_Z \circ f_* = f_* \circ \Gamma_{f^{-1}(Z)}$. and since f is proper on $\mathrm{supp}(\mathcal{F})$, $(f_*\mathcal{F})_y = \Gamma(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)})$. Thus, $R\Gamma_{\{\psi \geq 0\}}(Rf_*(\mathcal{F}))_y = 0$.

V. deeper results

   A. Involutivity Theorem (KS) - $SS(\mathcal{F})$ is a cosiotropic subset.

   B. $SS(\mathcal{F})$ is Lagrangian is equivalent to $\mathcal{F}$ is constructible (KS Thm 8.4.2)