TALK 4 : CONSTRUCTIBLE SHEAVES AND GENERATION

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These notes are for the fourth talk of participating topology seminar at UCLA, Spring 19. The main purpose of these notes is to explain Sections 4.1–3 of [3]. For more precise arguments, one should consult more references including [1], [2], and [4].

The main theorem of the working seminar is the following:

Theorem 1 (Nadler-Zaslow correspondence.)**.** Let X be a smooth, closed, real analytic n-manifold. There is an A_∞-quasi-embedding

$$\mu_X : Sh_c(X) \hookrightarrow TwFuk(T^*X),$$

which is a quasi-equivalence.

Here, the domain $Sh_c(X)$ of μ_X is the category of sheaves with bounded, constructible cohomology.

The main topic of this talk is to explain that every object of $Sh_c(X)$ can be generated from simple objects which are called standard objects. More precisely, we would like to sketch a proof of the following:

Theorem 2 ([3], Proposition 4.3.1)**.** Any object of $Sh_c(X)$ is isomorphic to one obtained from shifts of standard objects by iteratively forming cones. The same is true for costandard objects.

This talk consists of three parts. In the first part, we will define standard and costandard objects. In the second part, we will discuss the main difficulties of proving Theorem 2. Also, we will introduce two types of distinguished triangles, which help us to remove the main difficulties. Finally, we will sketch a proof of Theorem 2 in the last part.

1. Definitions

Through out these notes, let X be a smooth, closed real analytic n-dimensional manifold. Let Y be a submanifold of X, equipped with the subspace topology, satisfying $\bar{Y} \subset X$ and $\partial Y = \bar{Y} - Y \subset X$.

Let $i : Y \hookrightarrow X$ be the inclusion map on Y. Then, there is the (derived) direct image functor $i_* : Sh_c(Y) \to Sh_c(X)$ and the (derived) direct image with compact support functor $i_! : Sh_c(Y) \to Sh_c(X)$.

On Y, let $L_Y \in \mathrm{mod}(\mathbb{C}_Y)$ be a locally constant sheaf, where $\mathrm{mod}(\mathbb{C}_Y)$ is the category of sheaves on Y. We can construct a complex of sheaves on Y, which is concentrated at degree 0, as follow:

$$\cdots \to 0 \to L_Y \to 0 \to \cdots .$$

The constructed complex will be denoted L_Y again. Then, L_Y is an object of $Sh_c(Y)$.

We call the complex of sheaves $i_* L_Y$ a *standard object*. Similarly, $i_! L_Y$ is called a *costandard object*.

Remark 1.1. As implied by their names, standard and costandard objects have some relation. The Verdier dual functors $\mathcal{D}_X$ and $\mathcal{D}_Y$ intertwine those extensions. More precisely, we have the following equivalence

$$\mathcal{D}_X(i_! L_Y) \simeq i_* \mathcal{D}_Y(L_Y).$$

2. Distinguished triangles

Difficulties : Theorem 2 claims, roughly speaking, every object of $Sh_c(X)$ can be obtained from standard objects by iteratively forming cones. Thus, the main difficulties of proving Theorem 2 are coming from the differences between usual objects of $Sh_c(X)$ and standard objects.

Remark 2.1. From now on, we will assume that an object $\mathcal{F}^\bullet \in Sh_c(X)$ is a bounded complex of constructible sheaves, since every object of $Sh_c(X)$ has a bounded, constructible cohomology. Moreover, we can assume that there is a stratification $\mathcal{S}$ such that $\mathcal{F}^n$ is $\mathcal{S}$-constructible for all $n \in \mathbb{Z}$.

The first difference between usual objects of $Sh_c(X)$ and standard objects is that a standard object has a submanifold $Y \subset X$ such that the standard object is locally constant on Y. However, usual objects of $Sh_c(X)$ are locally constant on each stratum of $\mathcal{S}$. Thus, we will consider standard objects which are locally constant on each stratum. Then, we will need to combine those standard objects together. In other words, the first problem is how to combine them by iteratively forming cones.

The second difference is that a standard object is concentrated at degree 0, but an object $\mathcal{F}^\bullet$ of $Sh_c(X)$ is not necessarily to be concentrated at degree 0. Thus, the second problem is how to obtain an usual complex from complexes concentrated at a degree by iteratively forming cones.

To handle those problems, we need two types of distinguished triangles.

The first type of distinguished triangles : From now on, we will discuss a type of distinguished triangles, which helps us to solve the first problem.

Let Y be a closed submanifold of X. We define two inclusions

$$j : Y \hookrightarrow X,$$
$$i : Y^c = X - Y \hookrightarrow X.$$

Then, for any object $\mathcal{F}^\bullet \in Sh_c(X)$, there is a distinguished triangle

$$j_* j^! \mathcal{F}^\bullet \to \mathcal{F}^\bullet \to i_* i^{-1} \mathcal{F}^\bullet \xrightarrow{[1]} . \tag{2.1}$$

Because j is a closed inclusion, we can define the exceptional image functor $j^!$ on $\mathrm{mod}(\mathbb{C}_X)$, instead of $Sh_c(X)$.

Definition 2.2. The exceptional inverse image functor $j^! : \mathrm{mod}(\mathbb{C}_X) \to \mathrm{mod}(\mathbb{C}_{\mathbb{Y}})$ of a closed inclusion j is defined as follow: for every $\mathcal{F} \in \mathrm{mod}(\mathbb{C}_X)$ and every open subset $U \subset Y$,

$$(j^! \mathcal{F})(U) = \{s \in \mathcal{F}(V) \mid \mathrm{supp}(s) \subset Y\}, \tag{2.2}$$

where V is an open subset of X such that $V \cap Y = U$.

Remark 2.3. Definition 2.2 does not depend on the choice of an open set $V \subset X$.

Thus, if we are lucky, we could get a short exact sequence

$$0 \to j_* j^! \mathcal{F} \xrightarrow{f} \mathcal{F} \xrightarrow{g} i_* i^{-1} \mathcal{F} \to 0, \tag{2.3}$$

instead of Equation (2.1). From now on, we will explain Equation (2.3).

Intuitively, Equation 2.3 could be understood easier after feeding an open subset $U \subset X$, even though it does not give us a concrete proof of the exactness of (2.3). By definition,

$$(j_* j^! \mathcal{F})(U) = \{s \in \mathcal{F}(U) \mid \operatorname{supp}(s) \subset Y\},$$

for all open subset $U \subset X$. Thus, we have a natural inclusion

$$f_U : (j_* j^! \mathcal{F})(U) \hookrightarrow \mathcal{F}(U).$$

Similarly, by definition,

$$(i_* i^{-1} \mathcal{F})(U) = \mathcal{F}(U \cap Y^c).$$

Thus, there is the obvious restriction map $g_U : \mathcal{F}(U) \to \mathcal{F}(U \cap Y^c)$.

Then, first, f_U is injective because f_U is an inclusion. Next, it is easy to check that $f_U \circ g_U = 0$, since any $s \in (j_* j^! \mathcal{F})(U)$ has a support contained in Y. Similarly, if $s \in \ker(g_U)$, then for any $x \in U \cap Y^c$, $s_x = (g_U(s))_x = 0$. Thus, the $\operatorname{supp}(s)$ is disjoint to Y^c, in other words, is contained in Y. It means that $s \in \operatorname{Im}(f_U)$.

The last thing we should check is that $g_U : \mathcal{F}(U) \to \mathcal{F}(U \cap Y^c)$ is surjective for any open $U \subset X$. However, this is not always true. We might need to prove the surjectivity on the stalk levels.

To prove the exactness of Equation (2.3), it is enough to prove that the sequence of stalks

$$0 \to (j_* j^! \mathcal{F})_x \xrightarrow{f} \mathcal{F}_x \xrightarrow{g} (i_* i^{-1} \mathcal{F})_x \to 0 \tag{2.4}$$

is exact for every $x \in X$. Then, the following lemma completes the proof.

Lemma 1. Let $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \operatorname{mod}(\mathbb{C}_X)$. Then,

$$\mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$$

is exact at $\mathcal{G}$ if and only if

$$\mathcal{F}_x \xrightarrow{\phi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x$$

is exact at $\mathcal{G}_x$ for every $x \in X$.

Proof. The sequence of sheaves is exact if and only if $\operatorname{Im}(\phi) = \ker(\psi)$ as sheaves. It is equivalent to say that their stalks are the same for every $x \in X$, i.e.,

$$\operatorname{Im}(\phi)_x = \ker(\psi)_x.$$

In other words,

$$\operatorname{Im}(\phi_x) = \ker(\psi_x).$$

□

The stalks of $j_* j^! \mathcal{F}$ and $i_* i^{-1} \mathcal{F}$ are not easy to compute. For example, if $x \in Y^c$, then $(i_* i^{-1} \mathcal{F})_x = \mathcal{F}_x$, and if $x \in \operatorname{Int}(Y)$, then $(i_* i^{-1} \mathcal{F})_x = 0$. Note that $\operatorname{Int}(Y)$ is the interior of Y on X. However, if $x \in \partial Y = Y - \operatorname{Int}(Y)$, then it is not easy to compute $(i_* i^{-1} \mathcal{F})_x$.

We can assume that $\mathcal{F}$ is a $\mathcal{S}$-constructible sheaf. Also, we will assume that Y is the union of strata of $\mathcal{S}$ of dimension less that k for some $k \in \mathbb{N}$. Based on these assumptions, we can prove the exactness of Equation (2.4).

Remark 2.4. Similarly, we have another distinguished triangle

$$i_*i^!\mathcal{F}^\bullet \to \mathcal{F}^\bullet \to j_*j^{-1}\mathcal{F}^\bullet \overset{[1]}{\to} . \tag{2.5}$$

Equation (2.5) will be used for proving the second assertion of Theorem 2.

With Equation (2.1), we can obtain $\mathcal{F}^\bullet$ by forming the cone of $i_*i^{-1}\mathcal{F}^\bullet \overset{[1]}{\to} j_*j^!\mathcal{F}^\bullet$. Later, we will choose "nice" i and j, then $i_*i^{-1\bullet}$ and $j_*j^!\mathcal{F}^\bullet$ can be obtained from shifts of standard objects by iteratively forming cones.

The second type of distinguished triangles : From now on, we will discuss another type of distinguished triangles, which helps us to solve the second problem.

The type of distinguished triangles is associated to truncation functors. Thus, we will define the truncation functors first. Let $\tau_{\leq \ell}$ be a functor which assigns to $(\mathcal{F}^\bullet, d_\bullet)$ the truncated complex

$$\cdots \to \mathcal{F}^{\ell-1} \to \ker(d_\ell) \to 0 \to \cdots .$$

Similarly, let τ_ℓ be a functor which assigns to $(\mathcal{F}^\bullet, d_\bullet)$ another truncated complex

$$\cdots \to 0 \to \mathrm{Im}(d_\ell) \to \mathcal{F}^{\ell+1} \to \cdots .$$

Remark 2.5. In the above complexes, we understand $d_\ell : \mathcal{F}^\ell \to \mathcal{F}^{\ell+1}$ as a morphism between two sheaves. Then, $\ker(d_\ell)$ and $\mathrm{Im}(d_\ell)$ are kernel and image sheaves of d_ℓ.

Moreover, for every $n \in \mathbb{Z}$, there is a natural short exact sequence of sheaves,

$$0 \to \mathcal{F}^n \to \mathcal{F}^n \to 0 \to 0 \text{ if } n < \ell, \tag{2.6}$$

$$0 \to \ker(d_\ell) \to \mathcal{F}^\ell \to \mathrm{Im}(d_\ell) \to 0, \tag{2.7}$$

$$0 \to 0 \to \mathcal{F}^n \to \mathcal{F}^n \to 0 \text{ if } n > \ell, \tag{2.8}$$

Remark 2.6. It is easy to check the exactness of Equations (2.6)–(2.8) on the stalk levels. Then, Lemma 1 proves the exactness of the middle.

Equations (2.6)–(2.8) imply a distinguished triangle

$$\tau_{\leq \ell}\mathcal{F}^\bullet \to \mathcal{F}^\bullet \to \tau_{>\ell}\mathcal{F}^\bullet \overset{[1]}{\to} . \tag{2.9}$$

By forming the cone of $\tau_{>\ell}\mathcal{F}^\bullet \overset{[1]}{\to} \tau_{\leq \ell}\mathcal{F}^\bullet$, we obtain $\mathcal{F}^\bullet$. This helps us to solve the second problem.

3. Sketch of the proof of Theorem 2

We prove the first assertion of Theorem 2. The second assertion for costandard objects can be proved in a similar way.

Let $\mathcal{F}^\bullet \in Sh_c(X)$. We will assume that $\mathcal{F}^\bullet$ is a bounded complex of constructible sheaves. Then, there exists a stratification $\mathcal{S}$ of X such that $\mathcal{F}^n$ is $\mathcal{S}$-constructible for every $n \in \mathbb{Z}$. The proof is an induction on the strata, beginning with the open strata, i.e., the strata of the maximal dimension.

Let $\mathcal{S}_k$ be the union of the strata of dimension equal to k, and let $i_k : \mathcal{S}_k \hookrightarrow X$ be the inclusion of $\mathcal{S}_k$. Similarly, let $\mathcal{S}_{<k}$ be the union of the strata of dimension less than k, and let $j_{<k} : \mathcal{S}_{<k} \hookrightarrow X$ be the inclusion of $\mathcal{S}_{<k}$.

Suppose X has dimension equal to n. Then, the complement of $\mathcal{S}_n$ is $\mathcal{S}_{<n}$. Thus, we have the distinguished triangle

$$j_{<n*}j^!_{<n}\mathcal{F}^\bullet \to \mathcal{F}^\bullet \to i_{n*}i_n^{-1}\mathcal{F}^\bullet \xrightarrow{[1]}$$

of the first type. Then, $\mathcal{F}^\bullet$ is isomorphic to the cone of $i_{n*}i_*^{-1}\mathcal{F}^\bullet \xrightarrow{[1]} j_{<n*}j^!_{<n}\mathcal{F}^\bullet$.

Using distinguished triangles of the second type, we may express the sheaf $\mathcal{F}_n^\bullet := i_{n*}i_n^{-1}\mathcal{F}^\bullet$ by iteratively forming cones of shifted standard objects. Note that $i_n^{-1}\mathcal{F}^\ell$ is a locally constant sheaf on $\mathcal{S}_n$ for every $\ell \in \mathbb{Z}$, thus, $\mathcal{F}_n^\ell := i_{n*}i_n^{-1}\mathcal{F}^\ell$ is a (shifted) standard object which are associated to the strata $\mathcal{S}_n$. Moreover, by construction, the sheaf $\mathcal{F}^\bullet_{<n} := j_{<n*}j^!_{<n}\mathcal{F}^\bullet$ is supported on $\mathcal{S}_{<n}$.

Next, we have the distinguished triangle

$$j_{<n-1*}j^!_{<n-1}\mathcal{F} \to \mathcal{F} \to i_{n-1*}i_{n-1}^{-1}\mathcal{F} \xrightarrow{[1]}$$

on $\mathcal{S}_{<n}$. Again, using distinguished triangles associated to truncation functors, we express the sheaf $\mathcal{F}_{n-1} := i_{n-1*}i_{n-1}^{-1}\mathcal{F}_{<n}$ by iteratively forming cones of shifted standard objects associated to $\mathcal{F}_{n-1}$. Moreover, by construction, the sheaf $\mathcal{F}_{<n-1} := j_{<n-1*}j^!_{<n-1}\mathcal{F}$ is supported on $\mathcal{S}_{<n-1}$.

By repeating this procedure, we can prove that $\mathcal{F}$ may be expressed by iteratively forming cones of shifted standard objects. □

Nadler and Zaslow [3] proved the following stronger version of Theorem 2.

Theorem 3 ([3], Proposition 4.3.2)**.** Any object of $Sh_c(X)$ is isomorphic to one obtained from shifts of constand standard objects $i_*\mathcal{C}_U$ for open submanifold $i : U \hookrightarrow X$ by iteratively forming cones. The same is true for constand costandard objects $i_!\mathcal{C}_U$.

The authors proved Theorem 3 by choosing a "nice" stratification $\mathcal{T}$ such that all strata of $\mathcal{T}$ are cells. We skip the detailed proof of Theorem 3 in these notes.

References

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