Talk 3: Review of constructible sheaves and operations on sheaves

Good resources for the material presented today include [KS13], [Dim04], [Ive86], and [NZ09].

3.1 Constructible sheaves

Throughout this section, X will be a closed, real analytic manifold of dimension n, unless otherwise noted.

As indicated by the name, constructible sheaves are those that we can build from very simple pieces using relatively simple methods. We start with the simplest possible sheaf (of $\mathbb{C}$-modules) on X, which is the *constant sheaf* $\mathbb{C}_X$. This is the sheafification of

$$\mathbb{C}_X(U) = \mathbb{C}, \quad (V \hookrightarrow U) \mapsto (\mathbb{C} \xrightarrow{\mathrm{Id}} \mathbb{C})$$

for open sets $V \subset U \subset X$. Notice that for any other sheaf $\mathcal{F}$ on X we have an obvious map

$$\mathbb{C}_X \times \mathcal{F} \to \mathcal{F},$$

defined by the fact that $\mathcal{F}(U)$ is a $\mathbb{C}$-module for each $U \subset X$. This motivates the notation $\mathbf{mod}(\mathbb{C}_X)$ for sheaves on X. Also note that we have a constant sheaf $M_X \in \mathbf{mod}(\mathbb{C}_X)$ for each $M \in \mathbf{mod}(\mathbb{C})$.

Next, we allow ourselves to patch constant sheaves together, perhaps with monodromy. A *locally constant sheaf* is a sheaf $\mathcal{F} \in \mathbf{mod}(\mathbb{C}_X)$ such that each $x \in X$ admits an open neighborhood $U \subset X$ for which $\mathcal{F}|_U$ is a constant sheaf.

Example 4. Let $\pi \colon Y \to X$ be a finite-sheeted covering space with fiber F. We define a sheaf $\mathcal{F}$ by assigning

$$\mathcal{F}(U) = \mathbb{C}\langle \Gamma(\pi \colon Y \to U)\rangle,$$

with the obvious restriction maps. Given $x \in X$, we can choose an open neighborhood $U \subset X$ such that $\pi^{-1}(U) \simeq U$, so $\Gamma(\pi \colon Y \to U) \simeq F$. In particular,

$$\mathcal{F}|_U(V) = \mathbb{C}\langle F\rangle$$

for any $V \hookrightarrow U$, and $\mathcal{F}|_U$ is a constant sheaf. In fact, the category of locally constant sheaves is equivalent to the category of covering spaces on X (without the finite-sheeted requirement).

Finally, we allow ourselves to glue (finite rank) locally constant sheaves across stratifications. Before giving a precise definition, let's consider an example. Let $X = \Sigma S^1$ be the space depicted in Figure 1a, with the obvious stratification, and let $C \subset X$ be the stratum coming from the original S^1. Define a sheaf $\mathcal{F} \in \mathbf{mod}(\mathbb{C}_X)$ by declaring

$$\mathcal{F}(U) = \begin{cases} \mathbb{C}, & U \cap C \neq \emptyset \\ 0, & U \cap C = \emptyset \end{cases},$$

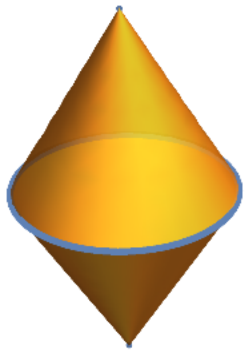
with restriction maps given by either the identity or the zero map in the obvious way. Notice that this sheaf is not locally constant: for any U which intersects C, $\mathcal{F}|_U$ cannot be constant, because $\mathcal{F}|_U(U) = \mathbb{C}$ and yet there exist open sets $V \subset U$ for which $\mathcal{F}|_U(V) = 0$. But the restriction $\mathcal{F}|_C$ to the 1-dimensional stratum is locally constant (indeed, this sheaf is constant). We say that $\mathcal{F}$ is *constructible* with respect to this stratification of X.

We'll now give a more formal definition of constructible sheaves on X. This requires first defining *stratifications*.

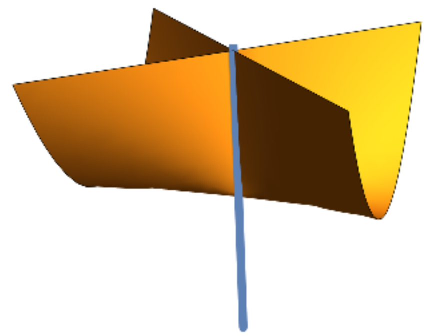
Definition. A C^p *stratification* of a real analytic manifold X is a locally finite covering $\mathcal{S} = \{S_\alpha\}$ of X by pairwise disjoint, locally closed C^p submanifolds $S_\alpha \subset X$ called *strata*, where the strata are required to satisfy

$$\overline{S}_\alpha \cap S_\beta \neq \emptyset \quad \text{if and only if} \quad S_\beta \subset \overline{S}_\alpha,$$

which is called the *axiom of the frontier*.



(a) ΣS^1.



(b) A Whitney umbrella.

Figure 1: A stratification and a filtration.

Some spaces (namely, algebraic varieties) admit natural filtrations which are not stratifications. For instance, consider the *Whitney umbrella*, defined by

$$W = \{(x, y, z) | x^2 = zy^2\} \subset \mathbb{R}^3$$

and depicted in Figure 1b. This variety admits a natural decomposition into the z-axis Z and a the smooth manifold $Y = W \setminus Z$. But notice that the axiom of frontier fails: $\overline{Y} \cap Z \neq \emptyset$, and yet Z is not a subset of $\overline{Y}$.

This is not the only peculiarity of the natural filtration of the Whitney umbrella. It's also the case the lower-dimensional stratum Z appears to be transverse to the higher-dimensional stratum Y. We don't allow ourselves to glue locally constant sheaves across such stratifications, so we rule this behavior out with *Whitney's condition*.

Definition. Let X, Y be C^1 submanifolds of a real-analytic manifold M. Consider the following scenario:

(1) $x \in X$;

(2) $x_i \in X$ and $y_i \in Y$ are sequences converging to x;

(3) the secant lines $\ell_i = \overline{x_i y_i}$ converge to a line ℓ;

(4) the tangent planes $T_{y_i} Y$ converge to a plane τ.

(Of course the last two items require a local coordinate chart.) We say that (Y, X) satisfies *Whitney's condition* if whenever we have the above scenario, $\ell \subset \tau$.

We call a stratification $\mathcal{S} = \{S_\alpha\}$ a *Whitney stratification* if each pair of strata (S_α, S_β) satisfies Whitney's condition.

Remark. One reason for wanting our stratifications to satisfy Whitney's condition is the following. Fix a point $x \in X$ and consider a neighborhood $N_x \subset X$. This neighborhood will be stratified, and Whitney's condition ensures that if $y \in X$ is some other point in the same connected component of the stratum containing x, then y will admit a neighborhood N_y with the same stratification. We won't prove this (or even make it precise) here, but consider the Whitney umbrella. A point on the lower z-axis has a neighborhood which is just an interval, while a point on the upper z-axis has a neighborhood which is not smooth, and which does not lie in a single component of the filtration of W. So in a stratification, these points would need to lie in distinct strata.

We may one last requirement of our stratifications, which is that each stratum must be a *subanalytic subset* of X. We won't define these sets here, but point out two important (and defining?) properties of the collection of subanalytic subsets. The first is that every set of the form $\{x \in X | f(x) > 0\}$ for some analytic function $f \colon X \to \mathbb{R}$ is subanalytic, and the second is that we have the following refinement proposition.

Proposition 3.1. *Let X be a real-analytic manifold, and p a positive integer. For every locally finite collection $\mathcal{A}$ of subanalytic subsets of X, there is a C^p Whitney stratification $\mathcal{S}$ with connected, subanalytic strata such that*

$$S \cap A \neq \emptyset \Rightarrow S \subset A$$

for any $A \in \mathcal{A}$ and $S \in \mathcal{S}$.

Example 5. The natural filtration on the Whitney umbrella is not a Whitney stratification (nor even a stratification), but can be refined to obtain a Whitney stratification. Specifically, we can take the (open) upper and lower portions of the z-axis to each be a stratum, as well as letting the origin be its own stratum.

Finally, we have

Definition. Let X be a real analytic manifold, $\mathcal{S} = \{S_\alpha\}$ a C^p Whitney stratification by subanalytic submanifolds $S_\alpha \subset X$. We say that a sheaf $\mathcal{F}$ on X is *$\mathcal{S}$-constructible* if $\mathcal{F}|_{S_\alpha}$ is locally constant and finite rank, for every α. We call $\mathcal{F}$ *constructible* if there is some such stratification $\mathcal{S}$ for which $\mathcal{F}$ is $\mathcal{S}$-constructible.

This gives us a full subcategory $\mathbf{mod}_c(\mathbb{C}_X) \subset \mathbf{mod}(\mathbb{C}_X)$. We also have a full subcategory $D_c(X) \subset D(X)$, but the objects of $D_c(X)$ are *not* complexes of constructible sheaves. Instead, a complex $\mathcal{F}^\bullet$ is constructible if each cohomology sheaf $H^i(\mathcal{F}^\bullet)$ is constructible.

3.2 Operations on sheaves

Again we fix a closed, real analytic manifold X of dimension n. In this section we will define a number of functors on $\mathbf{mod}(\mathbb{C}_X)$, with an eye towards obtaining the six standard operations on (some version of) the derived category $D(X)$.

3.2.1 HOM and tensor product

In the last talk we defined a morphism $\varphi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ of sheaves to be a natural transformation of functors. Concretely, this means that we have a morphism

$$\varphi_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$$

of $\mathbb{C}$-modules for each open set $U \subset X$. Using pointwise operations, we see that $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ is itself a $\mathbb{C}$-module. We can in fact construct a sheaf. We let $\mathcal{H}om(\mathcal{F}, \mathcal{G}) \in \mathbf{mod}(\mathbb{C}_X)$ be the sheaf arising from the association

$$U \mapsto \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U),$$

with the obvious restriction maps. So we have

$$\mathcal{H}om \colon \mathbf{mod}(\mathbb{C}_X)^{op} \times \mathbf{mod}(\mathbb{C}_X) \to \mathbf{mod}(\mathbb{C}_X).$$

We also have a tensor product

$$\otimes \colon \mathbf{mod}(\mathbb{C}_X) \times \mathbf{mod}(\mathbb{C}_X) \to \mathbf{mod}(\mathbb{C}_X)$$

defined by

$$(\mathcal{F} \otimes \mathcal{G})(U) = \mathcal{F}(U) \otimes_{\mathbb{C}} \mathcal{G}(U).$$

Notice that since tensor products commute with direct limits, we have $(\mathcal{F} \otimes \mathcal{G})_x = \mathcal{F}_x \otimes_{\mathbb{C}} \mathcal{G}_x$. The analogous statement need not be true for the stalks of $\mathcal{H}om$.

3.2.2 Direct and inverse images

So we have an assignment $X \mapsto \mathbf{mod}(\mathbb{C}_X)$, and we've seen that $\mathbf{mod}(\mathbb{C}_X)$ is a relatively nice category. Next we want to make the assignment itself nice. Given a map $f\colon X \to Y$ of real analytic manifolds, we will define some associated functors.

(1) **Direct image.** In the last talk we defined a presheaf to be a functor $\mathcal{F}\colon \mathrm{Op}(X)^{op} \to \mathbf{mod}(\mathbb{C})$, where $\mathrm{Op}(X)$ is the poset category of open subsets of X. Pushing this (pre)sheaf forward to a (pre)sheaf on Y is then easy. We have

$$\begin{tikzcd} \mathrm{Op}(Y)^{op} \arrow[r, "f"] \arrow[rr, bend left, "f_*\mathcal{F}"] & \mathrm{Op}(X)^{op} \arrow[r, "\mathcal{F}"] & \mathbf{mod}(\mathbb{C}) \end{tikzcd}$$

A more reasonable way of writing this is to say that we have

$$(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$$

for each open set $V \subset Y$, with the obvious restriction maps.

(2) **Inverse image.** Just as we can push a sheaf forward along a map $f\colon X \to Y$, we can pull sheaves back. We define the *inverse image functor* f^{-1} by defining $f^{-1}\mathcal{G}$ to be the sheaf associated to the presheaf

$$U \mapsto \varinjlim_{V \supset f(U)} \mathcal{G}(V),$$

for every $\mathcal{G} \in \mathbf{mod}(\mathbb{C}_Y)$ and open set $U \subset X$.

Example 6. Consider the map $f\colon X \to \{*\}$. A sheaf over $\{*\}$ is just a module, and for any sheaf $\mathcal{F} \in \mathbf{mod}(\mathbb{C}_X)$, we see that $f_*\mathcal{F} = \Gamma(X, \mathcal{F})$. That is, $f_*\mathcal{F}$ is the module of global sections of $\mathcal{F}$. A module $M \in \mathbf{mod}(\mathbb{C}_{\{*\}})$ pulls back to $f^{-1}M = M_X$.

On the other hand we have $i\colon \{x\} \hookrightarrow X$. Then for any $M \in \mathbf{mod}(\mathbb{C})$, i_*M is a skyscraper sheaf, while $i^{-1}\mathcal{F} = \mathcal{F}_x$ for any $\mathcal{F} \in \mathbf{mod}(\mathbb{C}_X)$.

Fact. $(f \circ g)_* = f_* \circ g_*$ and $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Proposition 3.2. *Let $f\colon X \to Y$ be a morphism of real analytic manifolds. For sheaves $\mathcal{F} \in \mathbf{mod}(\mathbb{C}_X)$ and $\mathcal{G} \in \mathbf{mod}(\mathbb{C}_Y)$ we have a bijection*

$$\mathrm{Hom}_{\mathbf{mod}(\mathbb{C}_X)}(f^{-1}\mathcal{G}, \mathcal{F}) \simeq \mathrm{Hom}_{\mathbf{mod}(\mathbb{C}_Y)}(\mathcal{G}, f_*\mathcal{F}).$$

Proof. We follow the proof of [Ive86, II.4]. Consider the map $\varphi \mapsto \psi$. Here $\varphi \in \mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F})$ and its image $\psi \in \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F})$ is defined by the diagram

$$\begin{tikzcd} \mathcal{G}(V) \arrow[r, "\psi_V"] \arrow[d] & (f_*\mathcal{F})(v) \arrow[d, equal] \\ (f^{-1}\mathcal{G})(f^{-1}(V)) \arrow[r, "\varphi_{f^{-1}(V)}"] & \mathcal{F}(f^{-1}(V)) \end{tikzcd} \tag{1}$$

for each open subset $V \subset Y$. We claim that this is a bijection, essentially because morphisms of sheaves are determined by the morphisms they induce on stalks. For a point $x \in X$, take the direct limit of (1) over open sets $V \subset Y$ containing $f(x)$. Then we have

$$\begin{tikzcd} \mathcal{G}_{f(x)} \arrow[r, "\psi_{f(x)}"] \arrow[d, "\cong"] & (f_*\mathcal{F})_x \arrow[d, "b_x"] \\ (f^{-1}\mathcal{G})_x \arrow[r, "\varphi_x"] & \mathcal{F}_x \end{tikzcd}$$

for some morphism of $\mathbb{C}$-modules b_x which is independent of φ and ψ. Since $\varphi_x = b_x \circ \psi_{f(x)}$, we see that the stalks of ψ determine those of φ, and thus our map is injective. On the other hand, we can use ψ to define a morphism

$$\prod_{x \in U} \left(b_x \circ \psi_{f(x)} \colon \mathcal{G}_{f(x)} \to \mathcal{F}_x\right)$$

for each open set $U \subset X$ and check that this induces a morphism $(f^{-1}\mathcal{G}) \to \mathcal{F}$ of sheaves. That is, the stalk morphisms $b_x \circ \psi_{f(x)}$ assemble to a morphism of sheaves, and this proves surjectivity. □

Fact. This adjunction also holds at the level of sheaves:

$$f_* \mathcal{H}om(f^{-1}\mathcal{G}, \mathcal{F}) = \mathcal{H}om(\mathcal{G}, f_*\mathcal{F}) \in \mathbf{mod}(\mathbb{C}_Y).$$

3.2.3 Direct image with compact support

We observed above that the direct image functor can be used to identify the global sections of $\mathcal{F}$. Sometimes we only want the compactly supported sections of $\mathcal{F}$, where the *support* of a section $s \in \mathcal{F}(U)$ is the closed subset

$$\operatorname{supp}(s) = \{x \in U | s_x \neq 0\} \subset U.$$

Here s_x is the germ of s at $x \in U$. This motivates our third operation:

(3) **Direct image with compact support.** Given $\mathcal{F} \in \mathbf{mod}(\mathbb{C}_X)$ and an open set $V \subset Y$, set

$$(f_!\mathcal{F})(V) = \{s \in (f_*\mathcal{F})(V) | f|_{\operatorname{supp}(s)} \colon \operatorname{supp}(s) \to V \text{ is proper}\}.$$

Notice that if f is a proper map, then $f_! = f_*$.

Example 7. If $f \colon X \to \{*\}$ is the map to a point, then $f_!\mathcal{F} = \Gamma_c(X, F)$ is the module of compactly supported sections of $\mathcal{F}$.

3.2.4 Exceptional inverse image

There is one last operation we want to discuss, but we can't define it at the sheaf level — we must pass to the derived setting. To set this up, consider the adjunction of f^{-1} and f_* that we noticed above. These are additive functors between $\mathbf{mod}(\mathbb{C}_X)$ and $\mathbf{mod}(\mathbb{C}_Y)$, and we can define them as functors between $C(X)$ and $C(Y)$ by applying them level-wise. These functors interact well with chain homotopies, giving us functors between $K(X)$ and $K(Y)$. Finally, because f^{-1} and f_* are exact and left-exact, respectively, we have an adjunction

$$\begin{tikzcd} D^+(X) \arrow[d, shift left, "Rf_*"] \\ D^+(Y) \arrow[u, shift left, "f^{-1}"] \end{tikzcd},$$

where $Rf_* \colon D^+(X) \to D^+(Y)$ is the right derived functor of f_*. (Because f^{-1} is exact, it respects quasi-isomorphism, and therefore doesn't need to be derived.)

It turns out that if this adjunction were reversed, we could obtain (some form of) Poincaré duality as a corollary. Indeed, the following result gives us the desired adjunction.

Theorem 3.3. *(Verdier duality) Let $f \colon X \to Y$ be a morphism of real analytic manifolds. Then $Rf_! \colon D^+(X) \to D^+(Y)$ admits a right adjoint $f^! \colon D^+(Y) \to D^+(X)$. In $D^+(\mathbf{mod}(\mathbb{C}))$ we have*

$$R\operatorname{Hom}(Rf_!\mathcal{F}^\bullet, \mathcal{G}^\bullet) \cong R\operatorname{Hom}(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet),$$

where $R\operatorname{Hom}$ is the derived functor of Hom *(not $\mathcal{H}om$).*

This theorem guarantees the existence of our fourth operation:

(4) **Exceptional inverse image.** This is the functor $f^!: D^+(Y) \to D^+(X)$, right adjoint to $Rf_!$, whose existence is guaranteed by Verdier duality.

Remark. Because f_* is not generally right-exact, we can't find a right adjoint for f_* before passing to the derived category.

We won't prove Verdier duality in general, but we'll follow [Ive86] again in discussing a case where $f^!$ is in fact a derived functor. Suppose

$$f: W \hookrightarrow X$$

is the inclusion of a locally closed subspace $W \subset X$. That is, W is a closed subset of an open subset of X. If $\mathcal{F} \in \mathbf{mod}(\mathbb{C}_W)$ is a sheaf on W, then we see that $(f_!\mathcal{F})(U) \subset \mathcal{F}(U \cap W)$ for each open set $U \subset X$. In particular, one can check that

$$(f_!\mathcal{F})_x = \begin{cases} \mathcal{F}_x, & x \in W \\ 0, & x \notin W \end{cases}.$$

This allows us to prove the following equivalence of categories.

Proposition 3.4. *If $f: W \hookrightarrow X$ is the inclusion of a locally closed subspace, then the functor $f_!: \mathbf{mod}(\mathbb{C}_W) \to \mathbf{mod}(\mathbb{C}_X)$ is an equivalence of categories between $\mathbf{mod}(\mathbb{C}_W)$ and the full subcategory of $\mathbf{mod}(\mathbb{C}_X)$ whose objects are*

$$\{\mathcal{F} \in \mathbf{mod}(\mathbb{C}_X) | \mathcal{F}_x = 0 \textit{ for all } x \notin W\}.$$

Moreover, the inverse functor is f^{-1}.

Proof. For each $\mathcal{F} \in \mathbf{mod}(\mathbb{C}_W)$, notice that $f^{-1}f_!\mathcal{F} = \mathcal{F}$. On the other hand, suppose $\mathcal{G} \in \mathbf{mod}(\mathbb{C}_X)$ has the property that $\mathcal{G}_x = 0$ whenever $x \notin W$. Then the obvious map $\mathcal{G} \to f_*f^{-1}\mathcal{G}$ can be factored as

$$\mathcal{G} \to f_!f^{-1}\mathcal{G} \to f_*f^{-1}\mathcal{G},$$

and this provides an isomorphism $\mathcal{G} \cong f_!f^{-1}\mathcal{G}$. ☐

Finally, we can construct $f^!$ in this special case. Given a sheaf $\mathcal{G} \in \mathbf{mod}(\mathbb{C}_X)$, we denote by $\mathcal{G}^W \in \mathbf{mod}(\mathbb{C}_X)$ the sheaf with

$$\mathcal{G}^W(U) = \{s \in \mathcal{G}(U) | \operatorname{supp}(s) \subseteq W\},$$

for all open subsets $U \subset X$. Now define $f^!\mathcal{G} = f^{-1}\mathcal{G}^W$. The proof of Proposition 3.4 yields an isomorphism $f_!f^!\mathcal{G} \simeq \mathcal{G}^W$, so the sections of $f_!f^!\mathcal{G}$ are those sections of $\mathcal{G}$ whose support is contained in W, and we have a monomorphism $f_!f^!\mathcal{G} \to \mathcal{G}$. Moreover, if $\mathcal{F} \in \mathbf{mod}(\mathbb{C}_X)$ is such that the stalk $\mathcal{F}_x$ is zero whenver $x \notin W$, then any morphism $\mathcal{F} \to \mathcal{G}$ may be factored through this monomorphism:

$$\begin{tikzcd} \mathcal{F} \arrow[rr, dashed] \arrow[dr] & & f_!f^!\mathcal{G} \arrow[dl] \\ & \mathcal{G} & \end{tikzcd}.$$

In particular, we have an isomorphism

$$\operatorname{Hom}(f_!\mathcal{F}, \mathcal{G}) \cong \operatorname{Hom}(f_!\mathcal{F}, f_!f^!\mathcal{G})$$

for any $\mathcal{F} \in \mathbf{mod}(\mathbb{C}_W)$. But Proposition 3.4 gives us an isomorphism

$$\operatorname{Hom}(f_!\mathcal{F}, f_!f^!\mathcal{G}) \cong \operatorname{Hom}(\mathcal{F}, f^!\mathcal{G}),$$

and thus $\operatorname{Hom}(f_!\mathcal{F}, \mathcal{G}) \cong \operatorname{Hom}(\mathcal{F}, f^!\mathcal{G})$, as desired.

3.2.5 Verdier duality implies Poincaré duality

We won't have time to discuss this during the seminar, but if we're willing to accept one more fact about $f^!$, then we can prove Poincaré duality as a consequence of Verdier duality.

Fact. If X is a manifold of dimension n and $\mathcal{C}^\bullet \in D^+(\mathbf{mod}(\mathbb{C}))$ is the complex with $\mathbb{C}$ in degree zero and 0 elsewhere, then $f^!\mathcal{C}^\bullet = \mathbb{C}_X[n]$, where $f\colon X \to \{*\}$ is the map to a point.

Theorem 3.5 (Poincaré duality)**.** *If X is a manifold of dimension n and $\mathcal{F} \in \mathbf{mod}(\mathbb{C}_X)$ is a sheaf, then*

$$H^{n-i}(X;\mathcal{F}) \cong (H^i_c(X;\mathcal{F}))^\vee$$

for all i, where $\vee$ denotes the dual.

Proof. With i fixed, let $\mathcal{F}^\bullet \in D^+(X)$ have $\mathcal{F}$ in degree i and 0 elsewhere. According to Verdier duality we have

$$R\operatorname{Hom}(Rf_!\mathcal{F}^\bullet, \mathcal{C}) \cong R\operatorname{Hom}(\mathcal{F}^\bullet, \mathbb{C}_X[n]).$$

But we've seen that for this particular map f, $f_! = \Gamma_c$, so $Rf_! = R\Gamma_c$, and the left side is

$$\begin{aligned} R\operatorname{Hom}(Rf_!\mathcal{F}^\bullet, \mathcal{C}^\bullet) &= R\operatorname{Hom}(R\Gamma_c\mathcal{F}^\bullet, \mathcal{C}^\bullet) = R\operatorname{Hom}(H^\bullet(R\Gamma_c\mathcal{F}^\bullet), \mathcal{C}^\bullet) \\ &= (H^0(R\Gamma_c\mathcal{F}^\bullet))^\vee = (H^i_c(X;\mathcal{F}))^\vee, \end{aligned}$$

where the second equality uses the fact that a cochain complex is equivalent to its cohomology in the derived category, the third uses the fact that $\mathcal{C}^\bullet$ is concentrated in degree zero, and the last uses the concentration of $\mathcal{F}^\bullet$ in degree i. At the same time, the right hand side of Verdier duality is

$$R\operatorname{Hom}(\mathcal{F}^\bullet, \mathbb{C}_X[n]) = R\operatorname{Hom}(\mathcal{F}[i], \mathbb{C}_X[n]) = H^{n-i}(X;\mathcal{F}),$$

and this produces the desired isomorphism. $\square$

References

[Dim04] Alexandru Dimca. *Sheaves in Topology*. Springer Science & Business Media, 2004.

[Ive86] Birger Iversen. *Cohomology of Sheaves*. Springer-Verlag, 1986.

[KS13] Masaki Kashiwara and Pierre Schapira. *Sheaves on Manifolds*, volume 292. Springer Science & Business Media, 2013.

[NZ09] David Nadler and Eric Zaslow. Constructible sheaves and the Fukaya category. *J. Amer. Math. Soc.*, 22(1):233–286, 2009.