SHEAVES, DERIVED CATEGORIES, AND DG CATEGORIES

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1. Introduction

These notes were written for a talk on some background material for a seminar on sheaves in symplectic topology. They cover the basic language of sheaves, derived categories, and dg categories, at least at level needed for the seminar. In particular, they are based on parts of sections 2.1 and 2.2 of [3]. For precise statements and definitions, details, and proofs, one should certainly consult more trustworthy references, such as [1] or [2].

Also, all of the following material (sheaves, derived categories, etc.) can be presented using much more general language. For the most part we will only consider definitions in the context of sheaves on a manifold.

2. Sheaves

Throughout, X will denote a real-analytic manifold, **Vect** is the category of complex vector spaces, and $\mathbf{Op}(X)$ is the category with objects open subsets of U and morphisms inclusions. Precisely,

$$\mathrm{Hom}_{\mathbf{Op}(X)}(U,V) = \begin{cases} pt & \text{if } U \hookrightarrow V \\ \emptyset & \text{otherwise} \end{cases}.$$

Definition 2.1. A **presheaf (of complex vector spaces) on** X is a functor $\mathcal{F} : \mathbf{Op}(X)^{op} \to \mathbf{Vect}$.

In more down to earth words, a presheaf is something which associates a vector space to any open set in X. Since $\mathcal{F}$ is a (contravariant) functor, we get a *restriction map* $\rho_U^V : \mathcal{F}(V) \to \mathcal{F}(U)$ for every inclusion $U \hookrightarrow V$. An element s of $\mathcal{F}(U)$ is a *section of* $\mathcal{F}$ over U. If $U \hookrightarrow V$ and s is a section of V, we will use the notation $s\,|_U := \rho_U^V(s)$.

A sheaf is a presheaf which is subject to some extra constraints, namely, that compatible sections can be uniquely glued together. Precisely,

Definition 2.2. A **sheaf** on X is a presheaf $\mathcal{F} : \mathbf{Op}(X)^{op} \to \mathbf{Vect}$ satisfying

i. If U_i is an open cover of U and $s, t \in \mathcal{F}(U)$ satisfy $s\,|_{U_i} = t\,|_{U_i}$ for all i, then $s = t$.
ii. If U_i is an open cover of U with $s_i \in \mathcal{F}(U_i)$ satisfying $s_i\,|_{U_i \cap U_j} = s_j\,|_{U_i \cap U_j}$, then there is an $s \in \mathcal{F}(U)$ such that $s\,|_{U_i} = s_i$ for all i.

The standard example of a sheaf is the one which associates to each open set U the space of continuous functions on U. Briefly, continuous functions defined on open sets which agree on overlaps can be uniquely glued together to give a continuous function on the whole space, hence axioms i. and ii. above are satisfied. An example of a presheaf which is not a sheaf is the functor L^1 on $\mathbb{R}^n$, which associates to any open set U the space of integrable measurable functions on U. This functor does not satisfy axiom ii., since gluing together locally integrable functions does not necessarily produce a globally integrable function.

A morphism of sheaves $\mathcal{F} \to \mathcal{G}$ is just a natural transformation between the functors.

As a final remark, the sheaves on X form an *abelian category*. Among many other things, this means that we can talk about kernels and cokernels of morphisms.

3. Derived Categories

In this section we define the derived category of sheaves on X. The following construction can be performed on any abelian category.

Step 1: The category of chain complexes.

Definition 3.1. Let $\mathbf{C}(X)$ be the category with the following data: the objects are (co)chain complexes of sheaves, and the morphisms between two complexes of sheaves $\mathcal{F}^n$ and $\mathcal{G}^n$ are the degree 0 chain maps. We denote the full subcategory of bounded complexes by $\mathbf{C}^b(X)$ and likewise $\mathbf{C}^+(X)$ and $\mathbf{C}^-(X)$ for the bounded below and bounded above complexes.

A simple operation on complexes is the *shift functor*: given a complex $(\mathcal{F}^n, d^n_{\mathcal{F}})$ and $k \in \mathbb{Z}$, define $(\mathcal{F}[k]^n, d^n_{\mathcal{F}[k]})$ by $\mathcal{F}[k]^n := \mathcal{F}^{n+k}$ and $d^n_{\mathcal{F}[k]} := (-1)^k d^{n+k}_{\mathcal{F}}$.

Another construction with complexes that will be important is the mapping cone of a morphism.

Definition 3.2. Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of complexes. The **mapping cone** of f is an object $M(f)$ in $\mathbf{C}(X)$ given by the following data: $M(f)^n := \mathcal{F}^{n+1} \oplus \mathcal{G}^n$ and

$$d^n_{M(f)} := \begin{pmatrix} -d^{n+1}_{\mathcal{F}} & 0 \\ f^{n+1} & d^n_{\mathcal{G}} \end{pmatrix}.$$

Note that there is a sequence

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{\begin{pmatrix} 0 \\ id_{\mathcal{G}} \end{pmatrix}} M(f) \xrightarrow{(\, id_{\mathcal{F}} \; 0 \,)} \mathcal{F}[1].$$

Next, we recall the notion of chain homotopy.

Definition 3.3. Two morphisms of complexes $f, g : \mathcal{F} \to \mathcal{G}$ are **chain homotopic** if there are maps $s^n : \mathcal{F}^n \to \mathcal{G}^{n-1}$ such that $f^n - g^n = s^{n+1} \circ d^n_{\mathcal{F}} + d^{n-1}_{\mathcal{G}} \circ s^n$. A **chain homotopy equivalence** is a morphism f such that there is a morphism g in the opposite direction with $f \circ g$ and $g \circ f$ both chain homotopic to the identity.

Step 2: The homotopy category.

Next, we pass to the homotopy category of sheaves on X by considering chain homotopy classes of morphisms between complexes.

Definition 3.4. Let $\mathbf{K}(X)$ be the category with the following data: $Ob(\mathbf{K}(X)) := Ob(\mathbf{C}(X))$ and

$$\mathrm{Hom}_{\mathbf{K}(X)}(\mathcal{F}, \mathcal{G}) := \mathrm{Hom}_{\mathbf{C}(X)}(\mathcal{F}, \mathcal{G}) \,/ \sim$$

where $f \sim g$ if f and g are chain homotopic. Define $\mathbf{K}^b(X)$ by using $\mathbf{C}^b(X)$ instead, and likewise for $\mathbf{K}^-(X)$ and $\mathbf{K}^+(X)$.

Another thing one can do in a category of complexes is take cohomology. This is possible roughly because $\mathbf{C}(X)$ is an abelian category, hence has kernels and cokernels. Precisely, $H^* : \mathbf{C}(X) \to \mathbf{C}(X)$ is a functor (the differentials in the complex $H^*(\mathcal{F})$ are all 0). So if $f : \mathcal{F} \to \mathcal{G}$ is a morphism of complexes, there is an induced morphism $H^*(f) : H^*(\mathcal{F}) \to H^*(\mathcal{G})$.

Definition 3.5. A chain map $f : \mathcal{F} \to \mathcal{G}$ in $\mathbf{C}(X)$ is a **quasi-isomorphism** if $H^*(f)$ is an isomorphism.

In particular, a chain homotopy equivalence is a quasi-isomorphism, Note that quasi-isomorphism is not an equivalence relation in general: the complex

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0 \to \cdots$$

is quasi-isomorphic to the complex

$$\cdots \to 0 \to 0 \to \mathbb{Z}/2 \to 0 \to \cdots$$

but there is no (nontrivial) map from the latter to the former.

Step 3: The derived category.

Using the notion of quasi-isomorphism, we can now pass to the derived category of sheaves on X. Roughly, we will formally invert all of the quasi-isomorphisms in $\mathbf{K}(X)$. This is analogous to the process of localizing a ring by a certain subset, where a-priori non-invertible elements become invertible.

Here we give a brief and nonconstructive description localizing categories for the sake of time, but one can be much more explicit about what morphisms in the localized category look like and how they behave.

For now, here is an unhelpful and approximate definition:

Definition 3.6. Let S be a family of morphisms in a category $\mathbf{C}$. A **localization of $\mathbf{C}$ by** S is a category $\mathbf{C}_S$ and a functor $Q : \mathbf{C} \to \mathbf{C}_S$ such that $Q(s)$ is an isomorphism for all $s \in S$, and any functor $F : \mathbf{C} \to \mathbf{A}$ satisfying the property that $F(s)$ is an isomorphism for all $s \in S$ factors uniquely through Q.

One can show that if a localization of a category exists, it is unique up to equivalence of categories.

Definition 3.7. The **derived category of sheaves on** X is the localization of $\mathbf{K}(X)$ by the set of quasi-isomorphisms: $\mathbf{D}(X) := \mathbf{K}(X)_S$ where S is the family of quasi-isomorphisms. $\mathbf{D}^b(X)$ is defined by using $\mathbf{K}^b(X)$, etc.

It is important to note that $\mathbf{K}(X)$ and $\mathbf{D}(X)$ are not abelian categories. One reason why abelian categories are nice is because one has results like the snake lemma, and hence the existence of long exact sequences arising from short exact sequences. Fortunately, $\mathbf{K}(X)$ (and $\mathbf{D}(X)$) are *triangulated*. We will not define what this means here, but the triangles are sequences of morphisms $\mathcal{F} \to \mathcal{G} \to \mathcal{H} \to \mathcal{F}[1]$ which are isomorphic in $\mathbf{K}(X)$ to some mapping cone triangle $\mathcal{F}' \xrightarrow{f} \mathcal{G}' \to M(f) \to \mathcal{F}[1]$. These play the role of short exact sequences, in that they yield long exact sequences.

4. DG Categories

Derived categories as in the previous section are not well-behaved, for various reasons (see [4]). One motivation for dg categories is to perform a similar construction but in a way that remedies some of these issues.

Definition 4.1. A **differential graded (dg) category C** is a category such that every morphism set has the structure of a differential graded $\mathbb{Z}$-module.

Explicitly, for any two objects A, B in a dg category we have a decomposition

$$\mathrm{Hom}(A,B) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}^n(A,B)$$

and a map $d : \mathrm{Hom}^n(A,B) \to \mathrm{Hom}^{n+1}(A,B)$ such that $d^2 = 0$.

Suppose our category is abelian, so that we can make sense of kernels and cokernels and thus cohomology. For any two objects A and B, $\mathrm{Hom}^*(A,B)$ is a cochain complex and so we can consider $Z^0(\mathrm{Hom}^*(A,B))$ and $H^0(\mathrm{Hom}^*(A,B))$. Moreover, we can define new categories $Z^0(\mathbf{C})$ and $H^0(\mathbf{C})$, the latter being the *homotopy category of* $\mathbf{C}$, by letting $Ob(Z^0(\mathbf{C})) = Ob(H^0(\mathbf{C})) = Ob(\mathbf{C})$ and

$$\mathrm{Hom}_{Z^0(\mathbf{C})}(A,B) = Z^0(\mathrm{Hom}^*(A,B)) \quad \text{and} \quad \mathrm{Hom}_{H^0(\mathbf{C})}(A,B) = H^0(\mathrm{Hom}^*(A,B)).$$

The dg category of primary interest to us is the dg category of chain complexes of sheaves on X. This a category $\mathbf{C}_{dg}(X)$ whose objects are (co)chain complexes, as in $\mathbf{C}(X)$. The morphisms in $\mathbf{C}_{dg}(X)$ are complexes of maps (not necessarily chain maps) of degree n, not just degree 0 chain maps. Given two complexes $\mathcal{F}$ and $\mathcal{G}$, the grading on $\mathrm{Hom}_{\mathbf{C}_{dg}(X)}(\mathcal{F},\mathcal{G})$ is clear. The differential d in the complex $\mathrm{Hom}^*(\mathcal{F},\mathcal{G})$ is defined by

$$d(f) = d_{\mathcal{G}} \circ f - (-1)^n f \circ d_{\mathcal{F}}$$

for $f \in \mathrm{Hom}^n(\mathcal{F},\mathcal{G})$. Note that

$$Z^0(\mathbf{C}_{dg}(X)) = \mathbf{C}(X)$$

and

$$H^0(\mathbf{C}_{dg}(X)) = \mathbf{K}(X).$$

The point of this section is that there is a dg category $\mathbf{Sh}(X)$ such that $H^0(\mathbf{Sh}(X)) = \mathbf{D}(X)$. For this we will give another unhelpful and imprecise definition.

Definition 4.2. Let $\mathbf{C}$ be a dg category and $\mathbf{D}$ a full dg subcategory. The **dg quotient $\mathbf{C}/\mathbf{D}$** is a dg category such that any dg functor $\mathbf{C} \to \mathbf{A}$ such that the induced map $H^0(\mathbf{C}) \to H^0(\mathbf{A})$ sends $\mathbf{B}$ to 0 factors through $\mathbf{A} \to \mathbf{A}/\mathbf{B}$.

Let $\mathbf{B}$ be the full subcategory of $\mathbf{C}_{dg}(X)$ of acyclic objects, i.e., objects with 0 cohomology. The *dg derived category of sheaves on* X, denoted $\mathbf{Sh}(X)$, is the dg quotient $\mathbf{C}_{dg}(X)/\mathbf{B}$. It is then a fact that $H^0(\mathbf{Sh}(X)) = \mathbf{D}(X)$.

References

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