The Conormal Torus is a Complete Knot Invariant

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1 Introduction

[Disclamer: the following notes are very rough and likely contain errors/misconceptions]

Given a knot $K \subset \mathbb{R}^3$ with inclusion map $i : K \to \mathbb{R}^3$, we have a short exact sequence of vector bundles on K with ranks 2, 3, and 1, respectively:

$$0 \longrightarrow N^*K \longrightarrow i^{-1}T^*\mathbb{R}^3 \longrightarrow T^*K \longrightarrow 0. \tag{1}$$

Here N^*K is the conormal bundle of K in $\mathbb{R}^3$: it consists of covectors of $T^*\mathbb{R}^3$ that lie over points in K and annihilate TK. Given a Riemannian metric g on $\mathbb{R}^3$, we can consider the sphere bundle $S^2 \to S(T^*\mathbb{R}) \to \mathbb{R}^3$ and the circle bundle $S^1 \to S(N^*K) \to K$. The total space $S(N^*K)$ is thus a torus living inside $S(T^*\mathbb{R})$; we will call this torus the *conormal torus* $T_K \subset S(T^*\mathbb{R}^3)$. We will denote points in $T^*\mathbb{R}^3$ by (x,p), so

$$(x,p) = \sum_i p^i dx_i|_x. \tag{2}$$

Note that $(T^*\mathbb{R}^3, d\tau)$ is a symplectic manifold, where τ, the tautological 1-form, is

$$\tau = -\sum_i p^i dx_i. \tag{3}$$

Define an inclusion map $S(T^*\mathbb{R}^3) \to T^*\mathbb{R}^3$. Note that $(S(T^*\mathbb{R}^3), \sigma^*\tau)$ is a contact manifold. Since N^*K is a Lagrangian submanifold of $T^*\mathbb{R}^3$ (in general, the conormal bundle N^*Y of a closed submanifold $Y \subset X$ is a Lagrangian submanifold of T^*X), it follows that the conormal torus T_K is a Legendrian submanifold of $S(T^*\mathbb{R}^3)$.

Theorem 1.1. *Let Y be a closed submanifold of X. Then the conormal bundle N^*Y of Y in X is a Lagrangian submanifold of $(T^*X, d\tau)$.*

Proof. The tautological 1-form τ satisfies $\tau_\alpha = \pi^*\alpha$, where $\pi : T^*X \to X$ is the projection and $\alpha \in T^*X$ is an arbitrary covector. We want to show that $j^*d\tau = 0$, where $j : N^*Y \to T^*X$ is the inclusion. Define an inclusion $i : Y \to X$ and a projection $p : N^*Y \to Y$; note that $\pi \circ j = i \circ p$. Then

$$j^*d\tau_\alpha = dj^*\pi^*\alpha = dp^*i^*\alpha = 0, \tag{4}$$

since $\alpha \in N^*Y$ and thus lies in the kernel of $i^* : i^{-1}T^*X \to T^*Y$. □

A *Legendrian isotopy* $f : T^2 \times [0,1] \to S(T^*M)$ from T_K to $T_{K'}$ is a smooth map such that $f_0(T^2) = T_K$, $f_1(T^2) = T_{K'}$, and $f_t(T^2)$ is Legendrian for all $t \in [0,1]$. Given a Legendrian isotopy f, we can use Gray's theorem to construct a contact isotopy $\phi_t : S(T^*\mathbb{R}^3) \to S(T^*\mathbb{R}^3)$ from T_K to $T_{K'}$ (ϕ_t is a contactomorphism for all t) that is trivial outside some open set containing the image of f

Theorem 1.2. *(Shende) Given knots $K, K' \subset \mathbb{R}^3$, if there is a Legendrian isotopy from T_K to $T_{K'}$ then K and K' are either isotopic or mirror. (Translating simple problem into harder problem?)*

Possible motivation: We can define a functor from the category of smooth manifolds to the category of contact manifolds that takes a smooth manifold X to $(S(T^*X)$ with its standard contact structure and a smooth map $f : X \to Y$ to $f^* : S(T^*Y) \to S(T^*X)$. We can ask what properties are preserved by this functor.

2 Outline or proof

2.1 Step 1

Let (M, g) be a Riemannian manifold. One can show: Given $\phi_t : S(T^*M) \to S(T^*M)$ a compactly supported contact isotopy, there is a corresponding family of equivalences of categories $\Phi_t : D(\mathrm{Sh}(M)) \to D(\mathrm{Sh}(M))$, which satisfy certain properties involving K and K'. (Note that this is trivially true, since we can take Φ_t to be the identity functor, if we don't include the additional properties.) Here $D(\mathrm{Sh}(M))$ is the derived category of sheaves of $\mathbb{Z}$-modules on M (in general, all sheaves will be sheaves of $\mathbb{Z}$-modules unless noted otherwise). (The functor Φ_t is obtained via a "Fourier-Mukai transform").

2.2 Step 2

Using the equivalence of categories from Step 1, plus the additional properties involving K and K', we can restrict the equivalence of categories to increasingly smaller full subcategories, ultimately showing that there is an equivalence of categories $\mathrm{Loc}(M - K) \to \mathrm{Loc}(M - K')$, which satisfies some additional properties. The restriction is accomplished by characterizing the subcategories in terms of properties that are clearly preserved by the equivalence. Here $\mathrm{Loc}(X)$ is the category of locally constant sheaves of Int-modules on X, also known as the category of "local systems" on X. It is not hard to show that there is an equivalence of categories $\mathrm{Loc}(X) \to \mathbb{Z}[\pi_1(X)] - \mathrm{Mod}$. Thus we obtain an equivalence of categories $\mathbb{Z}[\pi_1(M - K)] - \mathrm{Mod} \to \mathbb{Z}[\pi_1(M - K')] - \mathrm{Mod}$, which satisfies some additional properties. (The rings $\mathbb{Z}[\pi_1(M - K)]$ and $\mathbb{Z}[\pi_1(M - K')]$ are "Morita equivalent".)

2.3 Step 3

Using the additional properties of the equivalence of categories $\mathbb{Z}[\pi_1(M - K)] - \mathrm{Mod} \to \mathbb{Z}[\pi_1(M - K')] - \mathrm{Mod}$, one can show:

1. There is an isomorphism of rings $\mathbb{Z}[\pi_1(M - K)] \to \mathbb{Z}[\pi_1(M - K')]$. Note that we need the additional properties, since there are nonisomorphic rings that are Morita-equivalent. For example, the rings $\mathbb{Z}$ and $M_2(\mathbb{Z}) = \{2 \times 2 \text{ matrices of integers}\}$ are nonisomorphic rings that are Morita equivalent (the functor $\mathbb{Z} - \mathrm{Mod} \to M_2(\mathbb{Z}) - \mathrm{Mod}$, $M \mapsto M \oplus M$ gives the equivalence). The additional property that forces the isomorphism of the rings is the commutativity of the following diagram:

$$\begin{array}{ccc} R - \mathrm{Mod} & \longrightarrow & S - \mathrm{Mod} \\ \downarrow{\scriptstyle h_R} & & \downarrow{\scriptstyle h_S} \\ \mathbb{Z} - \mathrm{Mod} & \xrightarrow{=} & \mathbb{Z} - \mathrm{Mod}\,. \end{array} \tag{5}$$

   Here the downward arrows are forgetful functors, which are representable; for example $h_R = \mathrm{Hom}_{R-\mathrm{Mod}}(R, -)$. Intuitively, this condition keeps the modules from growing in size (in our example the functor $\mathbb{Z} - \mathrm{Mod} \to M_2(\mathbb{Z}) - \mathrm{Mod}$ takes $\mathbb{Z}$ to $\mathbb{Z} \oplus \mathbb{Z}$), which is sufficient to force the rings to be isomorphic.

   A rigorous argument is that the ring R can be recovered from the endomorphisms of the forgetful functor h_R, as can be understood as follows. In general $\mathrm{Nat}(F, F)$ is a monoid, and Yoneda's lemma states that, as sets, $\mathrm{Nat}(h_R, F)$ is in bijection with FR. Using the additional structure, we have that $\mathrm{Nat}(h_R, h_R)$ is a ring, and Yoneda's implies this ring is isomorphic to $h_R R = \mathrm{Hom}_{R-\mathrm{Mod}}(R, R) = R$.

2. There is an isomorphism of groups $f : \pi_1(M - K) \to \pi_1(M - K')$. Note: there nonisomorphic groups with isomorphic group rings (examples are known involving finite groups of very large order

   Note: if G is torsion-free and $\mathbb{Z}[G]$ has no nontrivial units (that is, $\mathbb{Z}[G]^\times = \{\pm g \mid g \in G\}$), then G can be recovered from $\mathbb{Z}[G]$: it is the quotient of the units $\mathbb{Z}[G]^\times$ by the group of torsion units, which is $\{\pm 1\}$.

   There is an open conjecture that the group ring of a nontrivial group has no nontrivial units. If G is left-orderable (G has a total order $\leq$ such that $a \leq b$ implies $ca \leq cb$), then it is torsion-free and $\mathbb{Z}[G]$ has no nontrivial units.

   It is known that knot groups are left-orderable, so from $\mathbb{Z}[\pi_1(M - K)] \to \mathbb{Z}[\pi_1(M - K')]$ it follows that $\pi_1(M - K) \cong \pi_1(M - K')$.

3. The group homomorphism f respects "peripheral subgroups", up to the signs of the longitude and meridian, which implies that K and K' are isotopic or mirror. In general, given a knot $K \subset \mathbb{R}^3$ we can define the "peripheral subgroup" $P_K \subset \pi_1(\mathbb{R}^3 - K)$, which is given by $\mathbb{Z} \cdot [m]$ if K is the unknot and $\mathbb{Z} \cdot [m] \oplus \mathbb{Z} \cdot [\ell]$ otherwise. Here m is a meridian and ℓ is a longitude (see picture). The generators $[m]$ and $[\ell]$ are not uniquely defined; we can also take as generators $([m'], [\ell']) = ([m]^{-1}, [\ell])$ or $([m'], [\ell']) = ([m], [\ell] \cdot [m]^n)$ for $n \in \mathbb{Z}$. For the unknot $[\ell] = [m]^n$. (Example: square knot and granny knot have isomorphic knot groups but are not isotopic.))

3 Locally constant sheaves

The *constant sheaf* on a topological space X associated to a $\mathbb{Z}$-module F, denoted $\underline{F}$, is the sheafification of the sheaf that assigns constant functions $U \to F$ to each open set U. The stalks are given by $\underline{F}_x = F$ for all $x \in X$.

A sheaf $\mathcal{F}$ on a topological space X is *locally constant* if there is an open covering $\{U_i\}$ of X such that $\mathcal{F}|_{U_i}$ is a constant sheaf for each i. The stalks are given by $\mathcal{F}_x = F$ for all $x \in X$, for some $\mathbb{Z}$-module F.

Given a topological space X, the following categories are all equivalent:

1. The category of covering spaces $\pi : \tilde{X} \to X$.

2. The category of locally constant sheaves of sets on X.

3. The category of $\pi_1(X, x_0)$-sets; that is, sets with an an action of $\pi_1(X, x_0)$.

The functor from the second category to the third is given by sending a sheaf $\mathcal{F}$ to the stalk $\mathcal{F}_{x_0}$, with the action of $\pi_1(X, x_0)$ on $\mathcal{F}_{x_0}$ given by the monodromy action of the sheaf.

The following categories are all equivalent:

1. The category of locally constant sheaves of $\mathbb{Z}$-modules on X (also called the category of "local systems" on X).

2. The category of $\mathbb{Z}[\pi_1(X, x_0)]$-modules.

3. The category of $\mathbb{Z}$-modules M with a $\pi_1(X, x_0)$ action $\rho : \pi_1(X, x_0) \to \mathrm{Aut}(M)$, where $\mathrm{Aut}(M)$ is the group of $\mathbb{Z}$-module automorphisms of M.

Given a $\mathbb{Z}$-module F and a homomorphism $\rho : \pi_1(X, x_0) \to \mathrm{Aut}(F)$, we can construct the corresponding locally constant sheaf $\mathcal{F}$ as follows. Let $\underline{F}$ be the constant sheaf with group F on the universal cover $\pi : \tilde{X} \to X$. Define $\mathcal{F}$ such that

$$\mathcal{F}_\rho(U) = \{s \in \underline{F}(\pi^{-1}(U)) \mid s \circ \phi_g = \rho(g)s\}. \tag{6}$$

Here $\phi_g \in \mathrm{Aut}(\pi) \cong \pi_1(X, x_0)$ is the deck transformation $\phi_g : \tilde{X} \to \tilde{X}$ corresponding to the element $g \in \pi_1(X, x_0)$. We are viewing sections $s \in \underline{F}(V)$ as locally constant functions $s : V \to F$ for V an open subset of $\tilde{X}$. Note we are considering sections that are equivariant with respect to the actions of $\pi_1(X, x_0)$ on $\tilde{X}$ and F.

Note: local systems are analogous to vector bundles with a flat connection, and the way we have defined $\mathcal{F}_\rho(U)$ given a representation $\rho : \pi_1(X, x_0) \to \mathrm{Aut}(F)$ is analogous to the way one can define a flat connection ∇_ρ on a vector bundle $V \to E \to X$ corresponding to a representation $\rho : \pi_1(X, x_0) \to \mathrm{Aut}(V)$ starting from a the trivial connection on the trivial vector bundle $V \times \tilde{X} \to \tilde{X}$.

As an example, consider locally constant sheaves on S^1 with group $\mathbb{Z}$. Note that $\mathbb{Z}[\pi_1(S^1)] = \mathbb{Z}[x, x^{-1}]$, where x represents a single loop. Note that $\mathrm{Aut}(\mathbb{Z}) = \mathbb{Z}_2$.

3.1 Classification of locally constant sheaves

Note that local systems $\mathcal{F}$ with group M are classified via the Cech cohomology $\check{H}^1(X; \underline{\mathrm{Aut}(M)})$, note that $\underline{\mathrm{Aut}(M)}$ is generally *not* at sheaf of $\mathbb{Z}$-modules, since $\mathrm{Aut}(M)$ is generally nonabelian. Note that $\underline{\mathrm{Aut}(M)}$ can be nonabelian, but Cech cohomology $\check{H}^p(X; G)$ are defined when G is nonabelian if p is 0 or 1. In this case $\check{H}^1(X; G)$ does not have the structure of an abelian group, just the structure of a pointed set, where the point corresponds to a constant sheaf. Note that two representations $\rho, \rho' : \pi_1(X, x_0) \to \mathrm{Aut}(M)$ are equivalent if and only if they are conjugate, so we should have

$$\check{H}^1(X; \underline{\mathrm{Aut}(M)}) = \mathrm{Hom}(\pi_1(X, x_0), \mathrm{Aut}(F))/\{\text{conjugation}\}. \tag{7}$$

In terms of the category of $\mathbb{Z}[\pi_1(X,x_0)]$-modules, it seems that we can classify objects via the group cohomology $H^1_\phi(\pi_1(X,x_0);\mathrm{Aut}(M))$, where the group homomorphism $\phi:\pi_1(X,x_0)\to\mathrm{Aut}(M)$ is the trivial homomorphism that maps everything to the identity. Usually, one defines group cohomology $H(G,A)$ where A is a G-module (which means the same thing as a $\mathbb{Z}[G]$-module). But one can define group cohomology $H^p_\phi(G,A)$ for A nonabelian and $\phi:G\to A$ a group homomorphism if p is 0 or 1. The first cohomology set $H^1_\phi(G;A)$ is defined as follows. The 1-cocycles are functions $f:G\to A$ such that

$$f(gh)=f(g)\phi(g)f(h). \tag{8}$$

Two 1-cocycles f_1 and f_2 are equivalent if there is an $a\in A$ such that

$$af_1(g)=f_2(g)\phi(g)a. \tag{9}$$

Then $H^1_\phi(G;A)$ is defined to be the quotient set of the 1-cocycles by the equivalence relation. Based on this definition, we find that $H^1_\phi(\pi_1(X,x_0);\mathrm{Aut}(M))$ is given by the conjugacy classes of group homomorphisms $\rho:\pi_1(X,x_0)\to\mathrm{Aut}(M)$, and thus seems to give the correct classification; is there a better way to understand this? For abelian groups, can one relate group cohomology and Cech cohomology for higher p?

4 Microsupport of a sheaf

Let $\mathcal{F}^*\in D(\mathrm{Sh}(X))$ be a complex of sheaves on X. We define the *microsupport* $\mathrm{SS}(\mathcal{F}^*)$ of $\mathcal{F}^*$ to be a subset of T^*X, such that a covector $(x,p)\in T^*X$ is not in the microsupport if for all smooth functions ϕ on X such that $\phi(x)=0$ and $d\phi(x)=p$ we have an isomorphism

$$\varinjlim_{U\ni x} H^j(U;\mathcal{F}^*)\to\varinjlim_{U\ni x} H^j(U\cap\{\phi<0\};\mathcal{F}^*) \tag{10}$$

for all j.

A nonzero covector $(x,p)\in T^*X$ defines a half-plane $\tilde{H}_p=\{(x,v)\in T_xX\mid p(v)<0\}$ of the tangent plane T_xX (note that we require $p(v)<0$, rather than $p(v)>0$). Intuitively, this should correspond to a "half-plane" $H_p\subset X$ that is well-defined near p. For $p=0$ we have $\tilde{H}_p=\varnothing$ and $H_p=\varnothing$. We should thus be able to define a complex of stalks $\mathcal{F}^*_x$ and a complex of "microstalks" $\mathcal{F}^*_{(x,p)}$ by

$$\mathcal{F}^j_x=\varinjlim_{U\ni x}\mathcal{F}^i(U), \qquad \mathcal{F}^j_{(x,p)}=\varinjlim_{U\ni x}\mathcal{F}^j(U\cap H_p). \tag{11}$$

Note that $\mathcal{F}^j_{(x,0)}=0$, since $H_p=\varnothing$. (Warning: one needs to take some care with the case $p=0$; probably best to define "microstalks" only for $p\neq 0$.) We should then be able to formulate the microsupport condition in terms of a map $H^*(\mathcal{F}^*_x)\to H^*(\mathcal{F}^*_{(x,p)})$ (note that that taking stalks commutes with cohomology, since the functor that takes a sheaf to its stalk at a point is exact). For a sheaf $\mathcal{F}$, rather than a complex of sheaves $\mathcal{F}^*$, the microsupport condition should be expressed in terms of a map $\mathcal{F}_x\to\mathcal{F}_{(x,p)}$.

Given a $\mathbb{Z}$-module M and a topological space X, define $\underline{M}$ to be the constant sheaf on X, defined such that $\underline{M}(U)=M$ for any nonempty connected open set U.

Given a sheaf $\mathcal{F}$ and a set $V\subset\mathbb{R}$, define a sheaf $\mathcal{F}_V=j_!j^{-1}\mathcal{F}$, where $j:V\to\mathbb{R}$ is the inclusion.

The *support* of a sheaf $\mathcal{F}$ on a topological space X is the set of points with nonvanishing stalk:

$$\mathrm{Supp}(\mathcal{F})=\{x\in X\mid \mathcal{F}_x\neq 0\}. \tag{12}$$

Note that $\mathrm{Supp}(\mathcal{F})$ is not necessarily closed; see example in Section 4.2 below. (Warning: some authors (e.g. Kashiwara and Schapira) define the support of a sheaf to be the closure this.) The *support* of a section $s\in\mathcal{F}(U)$ of a sheaf $\mathcal{F}$ on a topological space X for an open set U of X is the set of points in U with nonvanishing germs:

$$\mathrm{Supp}(s)=\{x\in U\mid s_x\neq 0\}. \tag{13}$$

Here $s_x\in\mathcal{F}_x$ is the germ of s at x. Note that if $s_x=0$ then $s|_U$ is zero for some open neighborhood U of x, so $\mathrm{Supp}(s)$ is automatically closed in X.

Given a continuous map $f:X\to Y$ and a sheaf $\mathcal{G}$ on Y, we define the pullback sheaf $f^{-1}\mathcal{G}$ to be the sheafification of the presheaf defined such that for $U\subset X$ open we have

$$(f^{-1}\mathcal{G})(U)=\lim_{V\supset U}\mathcal{G}(V), \tag{14}$$

where V ranges over open sets in Y.

Note: the following example shows that we need to sheafify. Let $X = \mathbb{R}$ with the discrete topology, $Y = \mathbb{R}$ with the usual topology, $f : X \to Y$ the identity map on sets, and $\mathcal{G}$ the sheaf of continuous functions on Y. Define open subsets $U_0 = \mathbb{Q}$ and $U_1 = X - U_0$ on X. Then $(f^{-1}\mathcal{G})(U_0) = (f^{-1}\mathcal{G})(U_1) = \mathcal{G}(Y)$, and we cannot glue arbitrary sections in $(f^{-1}\mathcal{G})(U_0)$ and $(f^{-1}\mathcal{G})(U_1)$ to get a section in $(f^{-1}\mathcal{G})(X) = \mathcal{G}(Y)$.

Define the *direct image with compact support* functor $f_!$ such that

$$(f_!\mathcal{F})(V) = \{s \in \mathcal{F}(f^{-1}(V)) \mid f|_{\mathrm{Supp}(s)} : \mathrm{Supp}(s) \to V \text{ is proper}\}. \tag{15}$$

Note that $\mathrm{Supp}(s)$ is a closed subset of $f^{-1}(V) \subset X$. If f is proper then $f_! = f_*$. In general $f_!\mathcal{F}$ is a subsheaf of $f_*\mathcal{F}$.

4.1 Example: $V = [0, \infty)$

Note that $j : V \to \mathbb{R}$ is proper, so $j_! = j_*$. For any open set $U \subset \mathbb{R}$ we have that

$$\underline{\mathbb{Z}}_V(U) = (j_! j^{-1}\underline{\mathbb{Z}})(U) = \{\text{locally constant functions on } U \cap V\}. \tag{16}$$

The stalks of $\underline{\mathbb{Z}}_V$ are given by

$$(\underline{\mathbb{Z}}_V)_x = \begin{cases} \mathbb{Z} & \text{if } x \geq 0 \text{ (equivalently } x \in V), \\ 0 & \text{if } x < 0 \text{ (equivalently } x \notin V). \end{cases} \tag{17}$$

In particular, the stalk at $\partial V = \{0\}$ is nonzero: $(\underline{\mathbb{Z}}_V)_0 = \mathbb{Z}$. Note that $\mathrm{Supp}(\underline{\mathbb{Z}}_V) = V$ is closed. The "microstalks" of $\underline{\mathbb{Z}}_V$ are

$$(\underline{\mathbb{Z}}_V)_{(x,p)} = \begin{cases} \mathbb{Z} & \text{if } (x = 0, p < 0) \text{ or } (x > 0, p \neq 0) \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

Thus the singular support of $\underline{\mathbb{Z}}_V$ is

$$\mathrm{SS}(\underline{\mathbb{Z}}_V) = \{(0, p) \in T^*\mathbb{R} \mid p \geq 0\} \cup \{(x, 0) \in T^*\mathbb{R} \mid x > 0\}. \tag{19}$$

Note that the singular support of $\underline{\mathbb{Z}}_V$ points *inward* relative to V at 0. Note that

$$\mathrm{SS}(\underline{\mathbb{Z}}_V) \cap \{\text{zero section of } T^*\mathbb{R}\} = \overline{\mathrm{Supp}(\underline{\mathbb{Z}}_V)} = V. \tag{20}$$

4.2 Example: $V = (0, \infty)$

Note that $j : V \to \mathbb{R}$ is not proper. For any open set $U \subset \mathbb{R}$ we have that

$$\underline{\mathbb{Z}}_V(U) = (j_! j^{-1}\underline{\mathbb{Z}})(U) = \{\text{locally constant functions on } U \cap V \text{ that vanish near } 0\}. \tag{21}$$

By "vanish near 0" we mean, for example, that $\underline{\mathbb{Z}}_V((0, 1)) = 0$. Note also that

$$(j_* j^{-1}\underline{\mathbb{Z}})(U) = \{\text{locally constant functions on } U \cap V\}. \tag{22}$$

Thus, $j_! j^{-1}\underline{\mathbb{Z}}$ is a subsheaf of $j_* j^{-1}\underline{\mathbb{Z}}$. The stalks of $\underline{\mathbb{Z}}_V$ are given by

$$(\underline{\mathbb{Z}}_V)_x = \begin{cases} \mathbb{Z} & \text{if } x \in V \\ 0 & \text{if } x \notin V. \end{cases} \tag{23}$$

In particular, the stalk at $\partial V = \{0\}$ is zero: $(\underline{\mathbb{Z}}_V)_0 = 0$. Note that $\mathrm{Supp}(\underline{\mathbb{Z}}_V) = V$ is open. The "microstalks" of $\underline{\mathbb{Z}}_V$ are

$$(\underline{\mathbb{Z}}_V)_{(x,p)} = \begin{cases} \mathbb{Z} & \text{if } (x = 0, p < 0) \text{ or } (x > 0, p \neq 0) \\ 0 & \text{otherwise} \end{cases} \tag{24}$$

Thus the singular support of $\underline{\mathbb{Z}}_V$ is

$$\mathrm{SS}(\underline{\mathbb{Z}}_V) = \{(0, p) \in T^*\mathbb{R} \mid p \leq 0\} \cup \{(x, 0) \in T^*\mathbb{R} \mid x > 0\}. \tag{25}$$

(Warning: some care is needed to show that $(0, 0) \in \mathrm{SS}(\underline{\mathbb{Z}}_V)$.) Note that the singular support of $\underline{\mathbb{Z}}_V$ points *outward* relative to V at 0. Note that

$$\mathrm{SS}(\underline{\mathbb{Z}}_V) \cap \{\text{zero section of } T^*\mathbb{R}\} = \overline{\mathrm{Supp}(\underline{\mathbb{Z}}_V)} = V. \tag{26}$$

4.3 Example: skyscraper sheaf $\mathbb{Z}_0$ on $\mathbb{R}$

The support of $\mathbb{Z}_0$ is given by $\mathrm{Supp}(\mathbb{Z}_0) = \{0\}$. The microsupport of $\mathbb{Z}_0$ is given by the fiber $T_0^*\mathbb{R} \subset T^*\mathbb{R}$ over $0 \in \mathbb{R}$.

5 Fourier-Mukai transform

Define $T^\circ X$ to be the complement of the zero section in T^*X.

Theorem 5.1. *(KS Prop. 7.2.1) Let $\phi_t : S(T^*M) \to S(T^*M)$ be a compactly supported contact isotopy for $t \in [0,1]$. Then there is a unique kernel $\mathcal{K} \in \mathrm{sh}(M \times M \times I)$ such that $\mathrm{SS}(\mathcal{K}_t)_{T^\circ M \times T^\circ M}$ is the graph of ϕ_t and $\mathcal{K}_0$ is the constant sheaf on the diagonal. The resulting Fourier-Mukai transform $\Phi_t : \mathrm{sh}(M) \to \mathrm{sh}(M)$ is an equivalence of categories for all t.*

For fixed t, consider the symplectomorphism $\phi_t : T^\circ\mathbb{R} \to T^\circ\mathbb{R}$ given by

$$\phi_t(x,p) = (x + \mathrm{sgn}(p)t, p). \tag{27}$$

(Note: ϕ_t is not compactly supported.) The symplectomorphism ϕ_t corresponds to a contactomorphism on $S(T^*\mathbb{R})$ associated with Reeb flow, equivalently geodesic flow for the standard metric $g = dx \otimes dx$. For $t > 0$,the graph of ϕ_t in $T^*\mathbb{R}^2$ is

$$\{((x, x+t),\ (p,p)) \mid x, p \in \mathbb{R} \text{ and } p > 0\} \cup \{((x, x-t),\ (p,p)) \mid x, p \in \mathbb{R} \text{ and } p < 0\}. \tag{28}$$

For $t \geq 0$, define a closed set $V_t \subset \mathbb{R}^2$ by

$$V_t = \{(x,y) \in \mathbb{R}^2 \mid |x - y| \leq t\}. \tag{29}$$

Define an inclusion $j : \mathrm{Int}(V_t) \to \mathbb{R}$, where $\mathrm{Int}(V_t)$ is the interior of V_t, and define the constant sheaf $\underline{\mathbb{Z}}_{\mathrm{Int}(V_t)} = j_! j^{-1}\underline{\mathbb{Z}}$, where $\underline{\mathbb{Z}}$ is the constant sheaf on $\mathbb{R}$ corresponding to the $\mathbb{Z}$-module $\mathbb{Z}$. We claim that the microsupport of $\underline{\mathbb{Z}}_{\mathrm{Int}(V_t)}$ is given by the graph of ϕ_t in $T^*\mathbb{R}^2$. (We want this graph to be Lagrangian, so we need to introduce some signs.)

The graph of the symplectomorphism is the microsupport of the kernel sheaf.

It seems that skyscraper sheaves are the analog of delta functions; can we use transforms of them to recover the kernel? Why is Φ_t invertible; it seems to be smearing out the support (e.g. the skyscraper sheaf at 0 smears to the locally constant sheaf with support on $[-t,t]$)?

$$\Phi_t(\mathcal{F}^*) = p_{2!}(\mathcal{K} \otimes p_1^{-1}\mathcal{F}^*), \qquad \Psi_t(\mathcal{G}^*) = p_{1*}\,\mathrm{Hom}(\mathcal{K}, p_2^!\mathcal{G}^*). \tag{30}$$

Here some of these functors are derived (for example, p_{1*} and $p_{2!}$ should be Rp_{1*} and $Rp_{2!}$), but we are suppressing this.

6 Details of Step 2 of proof

Define a full subcategory of $\mathrm{sh}(\mathbb{R}^3) = D(\mathrm{Sh}(\mathbb{R}^3))$:

$$\begin{aligned} \mathrm{sh}_{T_K}(\mathbb{R}^3) &= \{\mathcal{F}^* \in \mathrm{sh}(X) \mid \mathrm{SS}(\mathcal{F}^*) \subset N^*K \cup (\text{zero section of } T^*\mathbb{R}^3)\} && (31)\\ &= \{\mathcal{F}^* \in \mathrm{sh}(X) \mid \mathrm{SS}(\mathcal{F}^*) \subset T_K^*\mathbb{R}^3 \amalg T_{\mathbb{R}^3 - K}^*\mathbb{R}^3\}. && (32) \end{aligned}$$

We have the following theorem:

Theorem 6.1. *(KS Prop 8.4.1) Let $X = \amalg_{\alpha \in A} X_\alpha$ be a stratification of X and $\mathcal{F}^*$ be a complex of sheaves in $D(\mathrm{Sh}(X))$. Then the cohomology sheaves $H^j(\mathcal{F}^*)$ are constructible for all j (that is, $H^j(\mathcal{F}^*)|_{X_\alpha}$ are locally constant for all j and all $\alpha \in A$) if and only $\mathrm{SS}(\mathcal{F}^*) \subset \amalg_{\alpha \in A} T_{X_\alpha}^* X$, where $T_{X_\alpha}^* X$ is the conormal bundle of X_α in X.*

A special case of this theorem for the stratification $X = X$ that seems understandable: $H^j(\mathcal{F}^*)$ is locally constant for all j if and only if $\mathrm{SS}(\mathcal{F}^*) \subset T_X^* X$.

It follows that

$$\mathrm{sh}_{T_K}(\mathbb{R}^3) = \{\mathcal{F}^* \in \mathrm{sh}(X) \mid H^j(\mathcal{F}^*) \text{ is } S\text{-constructible for all } j\}, \tag{33}$$

where S is the stratification of $\mathbb{R}^3$ given by $\mathbb{R}^3 = (\mathbb{R}^3 - K) \amalg K$.

The microsupport of the transform is contained in the set-theoretic transform of the singular support:

$$\mathrm{SS}(\Phi_t(\mathcal{F}^*)) \subset p_2(\mathrm{SS}(\mathcal{K}_t) \cap p_1^{-1}(\mathrm{SS}(\mathcal{F}^*))). \tag{34}$$

Because of this fact, and the statement regarding the singular support of $\mathcal{K}_t$ in Theorem 5.1, the equivalence $\Phi_1 : \mathrm{sh}(\mathbb{R}^3) \to \mathrm{sh}(\mathbb{R}^3)$ restricts to an equivalence $\mathrm{sh}_{T_K} \to \mathrm{sh}_{T_{K'}}$. Define full subcategories of $\mathrm{sh}_{T_K}(\mathbb{R}^3)$:

$$\mathrm{loc}(K) = \{\mathcal{F}^* \in \mathrm{sh}_{T_K}(\mathbb{R}^3) \mid H^j(\mathcal{F}^*|_K) \text{ is a locally constant sheaf for all } j\}, \tag{35}$$

$$\mathrm{loc}(\mathbb{R}^3 - K) = \{\mathcal{F}^* \in \mathrm{sh}_{T_K}(\mathbb{R}^3) \mid H^j(\mathcal{F}^*|_{\mathbb{R}^3 - K}) \text{ is a locally constant sheaf for all } j\}. \tag{36}$$

Pick a point $p \in \mathbb{R}^3 - K$. Using a semiorthogonal decomposition of $\mathrm{sh}_{T_K}(\mathbb{R}^3)$, we can show that

$$\mathrm{loc}(K) = \{\mathcal{F}^* \in \mathrm{sh}_{T_K}(X) \mid H^*(\mathcal{F}^*)_p = 0\}. \tag{37}$$

Since the equivalence $\mathrm{sh}_{T_K} \to \mathrm{sh}_{T_{K'}}$ preserves cohomology (is this true, and if so why?) and stalks, it restricts to an equivalence $\mathrm{loc}(K) \to \mathrm{loc}(K')$. From the semiorthogonal decomposition of $\mathrm{sh}_{T_K}(\mathbb{R}^3)$, it follows that it also restricts to an equivalence $\mathrm{loc}(\mathbb{R}^3 - K) \to \mathrm{loc}(\mathbb{R}^3 - K')$. We have that

$$\mathrm{Loc}(\mathbb{R}^3 - K) = \{\mathcal{F}^* \in \mathrm{loc}(\mathbb{R}^3 - K) \mid H^j(\mathcal{F}^*)_p = 0 \text{ for } j \neq 0\}. \tag{38}$$

Since the equivalence $\mathrm{loc}(K) \to \mathrm{loc}(K')$ preserves cohomology (is this true, and if so why?) and stalks, it restricts to an equivalence $\mathrm{Loc}(\mathbb{R}^3 - K) \to \mathrm{Loc}(\mathbb{R}^3 - K')$.

6.1 Semiorthogonal decompositions

Definition 6.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two triangulated subcategories of $D(X)$. We say that $\mathcal{A}$ and $\mathcal{B}$ form a *semiorthogonal decomposition* of $D(X)$ if

1. $\mathrm{Hom}(A, B) = 0$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.
2. For every $E \in D(X)$ there is a unique exact triangle $A \to E \to B \to A[1]$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

It seems that we could define an analogous concept for abelian categories:

Definition 6.2. Let $\mathcal{A}$ and $\mathcal{B}$ be two triangulated subcategories of $\mathcal{E}$. We say that $\mathcal{A}$ and $\mathcal{B}$ form a *semiorthogonal decomposition* of $\mathcal{E}$ if

1. $\mathrm{Hom}(A, B) = 0$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.
2. For every $E \in \mathcal{E}$ there is a unique short exact sequence $0 \to A \to E \to B \to 0$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Example: take $\mathcal{E}$ to be the category of S-constructible sheaves of $\mathbb{Z}$-modules $\mathcal{F}$ on $\mathbb{R}^3$ for the stratification $\mathbb{R}^3 = (\mathbb{R}^3 - K) \amalg K$, such that $\mathcal{F}$ restricts to constant sheaves on $\mathbb{R}^3 - K$ and K. Any such sheaf $\mathcal{F}$ is specified by a homomorphism $\rho : B \to A$ between abelian groups A and B, such that for any nonemtpy contractible open set $U \subset \mathbb{R}^3$ we have

$$\mathcal{F}(U) = \begin{cases} B & \text{if } U \cap K \neq \varnothing, \\ A & \text{otherwise.} \end{cases} \tag{39}$$

Note that $\mathcal{F}|_{\mathbb{R}^3 - K} = \underline{A}$ and $\mathcal{F}|_K = \underline{B}$. Then $\mathcal{A}$ is the category of such sheaves with $\rho : B \to A$ given by $0 \to A$, and $\mathcal{B}$ is the category of such sheaves with $\rho : A \to B$ given by $B \to 0$. Note that we have the commutative diagrams

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & A, \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ B \arrow[r] & 0. \end{tikzcd} \tag{40}$$

Note: if we were to swap $\mathcal{A}$ and $\mathcal{B}$, the corresponding diagrams would not commute. Note that given a point $p \in \mathbb{R}^3 - K$, we have that $\mathcal{F}_p = 0$ for any $\mathcal{F} \in \mathcal{B}$.