NOTES FOR MATH 290F: SHEAVES AND SYMPLECTIC GEOMETRY

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1. Nadler-Zaslow correspondence

The goal of this quarter's working seminar is to work through enough background to understand the Nadler-Zaslow correspondence. The main references are:

(1) Nadler-Zaslow [NZ], "Constructible sheaves and the Fukaya category",
(2) Nadler [N], "Microlocal branes are constructible sheaves",
(3) Kashiwara-Shapira [KS], "Sheaves on manifolds",
(4) Viterbo [V], "An introduction to symplectic topology through sheaf theory",
(5) Shende [S], `https://math.berkeley.edu/ vivek/274.html`,
(6) Auroux [A], "A beginner's introduction to Fukaya categories".

Let X be a closed smooth manifold of real dimension n. In [NZ] the authors assume that X is real analytic; this figures in what we mean by a "constructible sheaf".

Theorem 1.0.1 (Nadler-Zaslow correspondence)**.** *There is an A_∞-quasi-embedding*

$$\mu_X : Sh_c(X) \hookrightarrow TwFuk(T^*X)$$

which is a quasi-equivalence.

The left-hand side is the dg category of constructible complexes of sheaves on X and the right-hand side is the A_∞-category of twisted complexes over (some version of) the Fukaya category of T^*X. A dg category is an A_∞-category with trivial higher operations. *A_∞-quasi-embedding* means it's an A_∞-functor which, on the level of cohomology, is a fully faithful embedding of the corresponding derived categories

$$H(\mu_X) : D_c(X) \hookrightarrow DFuk(T^*X).$$

Quasi-equivalence means that every object of $DFuk(T^*X)$ is isomorphic to an object coming from $D_c(X)$.

1.1. **Constructible sheaves.**

A *sheaf* $\mathcal{F}$ on X is a contravariant functor from the category of open sets of X with morphisms which are inclusions to the category Ab of abelian groups subject to some gluing axioms. In particular, $\mathcal{F}$ assigns an abelian group $\mathcal{F}(U)$ to each open set $U \subset X$ and a restriction morphism $\mathcal{F}(U) \to \mathcal{F}(V)$ for each $V \subset U$. (Think of $\mathcal{F}(U)$ as a set of functions on U.)

Let $\mathcal{S} = \{S_\alpha\}$ be a Whitney stratification of X. For example, $X = \mathbb{R}$ can be stratified by $S_0 = \{0\}$, $S_- = (-\infty, 0)$, $S_+ = (0, +\infty)$.

Definition 1.1.1. A sheaf $\mathcal{F}$ is *$\mathcal{S}$-constructible* if the restrictions of $\mathcal{F}$ to each S_α has finite rank and is locally constant. $\mathcal{F}$ is *constructible* if there exists a Whitney stratification $\mathcal{S}$ with respect to which $\mathcal{F}$ is $\mathcal{S}$-constructible.

Example. Let A_0, A_+, A_- be abelian groups and $A_0 \to A_+$, $A_0 \to A_-$ be fixed homomorphisms. Let $U \subset \mathbb{R}$ be an open interval. We can define an $\mathcal{S}$-constructible $\mathcal{F}$ on $\mathbb{R}$ by assigning $\mathcal{F}(U) = A_0$ if $U \supset S_0$; $\mathcal{F}(U) = A_+$ if $U \subset S_+$; $\mathcal{F} = A_-$ if $U \subset S_-$.

Let $Sh_c(X)$ be the dg category of constructible complexes of sheaves (over $\mathbb{C}$) on X. (To be explained in more detail in a later talk.)

Remark 1.1.2. We'll see later that the definition of an object in the dg or derived category of constructible complexes of sheaves is *not* a complex of constructible sheaves.

The following allows us to get a handle on $Sh_c(X)$:

Theorem 1.1.3. *Any object of $Sh_c(X)$ is isomorphic to one obtained from "standard objects" $i_*\mathbb{C}_U$ where $i : U \hookrightarrow X$ is an open submanifold and $\mathbb{C}_U$ is the constant sheaf on U, by iteratively taking shifts and cones.*

In other words, $i_*\mathbb{C}_U$ are the generators of $Sh_c(X)$. A priori U is an open manifold with stratified boundary.

1.2. **Fukaya category.** Recall that the cotangent bundle T^*X is a Liouville domain with a canonical 1-form $\lambda = \sum_i p_i dq_i$, where q_i are coordinates on X and p_i are the dual coordinates. In local coordinates $d\lambda = \sum_i dp_i dq_i$ and the Liouville vector field is $Y = \sum_i p_i \partial_{p_i}$. The unit cotangent bundle is

$$ST^*X = \{(p,q) \mid |p| = 1\} \subset T^*X,$$

where I leave the definition of $|\cdot|$ to your imagination. Since $Y \pitchfork ST^*X$, ST^*X is contact and T^*X minus the 0-section is the symplectization of ST^*X.

The objects of $Fuk(T^*X)$ are closed (= compact without boundary) Lagrangians and noncompact Lagrangians with conical (viewed in the cotangent bundle) ends. [This description is only a first-order approximation since we actually need local system data on the Lagrangians.] In terms of the symplectization picture, this means that the union of ends of a noncompact L is a half cylinder over a Legendrian $\Lambda \subset ST^*X$.

$\mathrm{Hom}(L, L')$ is generated by transverse intersections of L and $\phi(L')$, where ϕ is a small pushoff in the positive Reeb direction so that $L_0 := L$ and $L_1 := \phi(L')$ intersect in a compact part of T^*X, and the Floer differential counts "rigid" J-holomorphic strips $u : \mathbb{R} \times [0,1] \to T^*X$ such that

- $u(\mathbb{R} \times \{i\}) \subset L_i$, $i = 0, 1$,
- $\lim_{s\to+\infty}(\mathbb{R} \times \{s\}) = p$,
- $\lim_{s\to-\infty}(\mathbb{R} \times \{s\}) = q$,

where $p, q \in L_0 \cap L_1$. Here J is a suitably chosen almost complex structure on T^*X.

$Fuk(T^*X)$ is an A_∞-category, meaning it has higher composition maps satisfying compatibility conditions.

Examples. Some Lagrangians in T^*X.

(1) The zero 0-section and a fiber $T^*_x X$.
(2) Graphs of closed and exact 1-forms on X, written as ω or df.
(3) If $Y \subset X$ is a closed submanifold, then the *conormal bundle* $T^*_Y X$ is the set (x, α), $x \in X$, $\alpha \in T^*_x X$, such that $\alpha(T_x Y) = 0$. (If we have a Riemannian metric, the conormal bundle can be viewed as the normal bundle to Y in X.) Verify that $T^*_Y X$ is Lagrangian with conical ends. Also note that the 0-section is equal to $T^*_X X$ and $T^*_x X = T^*_{\{x\}} X$, hence the notation.

(4) Let $U \subset X$ be an open set with smooth boundary ∂U. We take the union of $T^*_U U$ (the zero section over U) and the positive part of $T^*_{\partial U} X$ (i.e., α such that $\alpha(n) > 0$ for n pointing out from U along ∂U). This Lagrangian is only piecewise smooth, but can be approximated by the graph of df, where $f : U \to \mathbb{R}_+$ and $f \to +\infty$ as we approach ∂U (strictly speaking, we probably want a smooth collar neighborhood $\partial U \times [0,1] \subset \overline{U}$ of ∂U such that f is constant on each $\partial \times \{t\}$). In particular, try to draw the Lagrangian for $U = (0,1) \subset \mathbb{R}$.

1.3. **Functor μ_X.** Assuming that $U \subset X$ is an open set with smooth boundary, the functor μ_X takes $i_* \mathbb{C}_U$ to df, as described in the examples. (Clearly there is a range of choices of df, which we will not worry about here.)

One computes that, for open subsets $i_0 : U_0 \hookrightarrow X$ and $i_1 : U_1 \hookrightarrow X$:

Lemma 1.3.1. *There is a canonical quasi-isomorphism*

$$\mathrm{Hom}_{Sh}(i_{0*}\mathbb{C}_{U_0}, i_{1*}\mathbb{C}_{U_1}) \simeq (\Omega(\overline{U}_0 \cap U_1, \partial U_0 \cap U_1), d).$$

Here $(\Omega(U,V), d)$ is the relative de Rham complex of differential forms on $U - V$ whose support lies in $X - V$ and whose cohomology computes $H^*(U,V)$.

Recall by Floer, given df_0 and df_1 defined on all of X, there exists an almost complex structure such that the Floer cochain complex $CF(df_0, df_1)$ is equivalent to the Morse cochain complex counting gradient trajectories of $f_0 - f_1$. We can apply the same considerations to df_0 and df_1 for U_0 and U_1 to show that the Morse cohomology computes $H^*(\overline{U}_0 \cap U_1, \partial U_0 \cap U_1)$.

Claim 1.3.2. *The de Rham model and the Morse cochain model are quasi-isomorphic, with the quasi-isomorphism induced by a dga morphism from de Rham to Morse.*

I don't know how to prove this, but seems motivated by Witten's "Supersymmetry and Morse theory" paper, where he considers a family of differentials $d_t := e^{ft} d e^{-ft}$, where f is a Morse function, adjoints $d_t^* := e^{ft} d^* e^{-ft}$, and Laplacians $\Delta_t = d_t d_t^* + d_t^* d_t$. This way we obtain a 1-parameter family relating harmonic forms (limit as $t \to 0$) to Morse critical points (limit as $t \to \infty$).

This gives a functor that maps Homs to Homs. For the A_∞-version we need to consider Morse flow trees as studied by Fukaya and Oh.