Homework 6

Due Wednesday, May 15.

(1) (20 pts) Prove that a short exact sequence $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ of chain complexes gives rise to a long exact sequence in homology:

$$\begin{aligned} \dots \longrightarrow\ & H_k(A) \xrightarrow{\phi_k} H_k(B) \xrightarrow{\psi_k} H_k(C) \xrightarrow{\delta_k} \\ \longrightarrow\ & H_{k-1}(A) \xrightarrow{\phi_{k-1}} H_{k-1}(B) \xrightarrow{\psi_{k-1}} H_{k-1}(C) \xrightarrow{\delta_{k-1}} \dots . \end{aligned}$$

(In class we verified some steps. You need to do the rest.)

(2) Let $A, B \subset X$ be subspaces and $C_n(A+B) \subset C_n(X)$ be the chains which are sums of chains in A and chains in B. Prove there is an exact sequence

$$0 \to C_n(A \cap B) \xrightarrow{\phi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A+B) \to 0.$$

(3) Using the Mayer-Vietoris sequence, compute the singular homology groups $H_i(S^2)$ for all i. Then compute $H_i(S^n)$ for all i, n.

(4) Prove that there is no retraction of D^n to its boundary $\partial D^n = S^{n-1}$. Hence prove that every continuous map $f : D^n \to D^n$ has a fixed point.

(5) Let $X = S^1 = \mathbf{R}/\mathbf{Z}$. If $\sigma : \Delta^1 = [0,1] \to S^1 = \mathbf{R}/\mathbf{Z}$ is a chain which maps $\sigma(x) = x$, then (i) show it is a cycle, and (ii) show that σ is homologous to $\sigma_1 + \sigma_2$, where $\sigma_1, \sigma_2 : [0,1] \to \mathbf{R}/\mathbf{Z}$ map $\sigma_1(x) = \frac{1}{2}x$ and $\sigma_2(x) = \frac{1}{2} + \frac{1}{2}x$.

(6) Using the Mayer-Vietoris sequence, compute the singular homology groups $H_i(T^2)$ for all i.

(7) Prove that if X is a compact metric space and $\mathcal{U} = \{U_\alpha\}$ is an open cover of X, then there is a number $\varepsilon > 0$ such that every open ball of radius ε is contained in some U_α.

(8) Hatcher, Section 2.2, Exercises 9 (worth 20 pts), 10.